Partial Differential Equations
Spring 2015

Assignment 13.2
Due April 23

Exercise 1. Consider the general second order equation

$$A(x)y'' + B(x)y' + (C(x) + \lambda D(x))y = 0. \tag{1}$$

If $\int \frac{B(x)}{A(x)}\,dx$ denotes an antiderivative of $\frac{B(x)}{A(x)}$, show that multiplication by the *integrating factor* $\mu(x) = \frac{1}{A(x)} \exp\left(\int \frac{B(x)}{A(x)}\,dx\right)$ transforms (1) into a Sturm-Liouville equation.

Exercises 2-4:

- **a.** Put the problem into Sturm-Liouville form. State p, q and r.
- **b.** Determine whether or not the problem is regular.
- **c.** Determine the weighted inner product associated with the problem.
- **d.** Show that eigenfunctions with different eigenvalues are orthogonal with respect to the inner product of part **c**.

Exercise 2.

$$x^2y'' + 3xy' + e^x y + \lambda y = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = 0.$$

Exercise 3.

$$xy'' + (1-x)y' + \lambda y = 0, \quad 0 < x < \infty,$$
$$y, y' \text{ bounded as } x \to 0^+,$$
$$y, y' \text{ bounded as } x \to \infty.$$

Exercise 4.

$$(2-x)y'' + y' - y + \lambda y = 0, \quad -1 < x < 1,$$
$$y(-1) - y'(-1) = 0, \quad y'(1) = 0.$$

Exercise 5. Given a Sturm-Liouville equation

$$(p(x)y')' + (q(x) + \lambda r(x)y) = 0 \tag{2}$$

define the associated second order linear differential operator by

$$Ly = -\frac{1}{r(x)}\left((p(x)y')' + q(x)y\right).$$

a. Show that (y, λ) is an eigenfunction/eigenvalue pair for (2) if and only if $Ly = \lambda y$.

b. If $\langle \cdot, \cdot \rangle$ denotes the inner product on the interval $[a, b]$ with respect to the weight function $r(x)$, show that for any (suitably differentiable) functions y_1 and y_2 one has

$$\langle Ly_1, y_2 \rangle = \langle y_1, Ly_2 \rangle + p(x)W(y_1, y_2)(x)\Big|_a^b.$$

This result is known as *Green's identity.* [*Suggestion:* Integrate by parts twice.]

c. If (y_1, λ_1) and (y_2, λ_2) are eigenfunction/eigenvalue pairs for (2) with $\lambda_1 \neq \lambda_2$, use Green's identity to rederive the relationship

$$\langle y_1, y_2 \rangle = \left.\frac{p(x)W(y_1, y_2)(x)}{\lambda_1 - \lambda_2}\right|_a^b.$$