An Introduction to Partial Differential Equations

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Partial Differential Equations
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Ordinary differential equations (ODEs)

These are equations of the form

$$F(x, y, y', y'', y''', \dots) = 0 \tag{1}$$

where:

- $y = y(x)$ is an (unknown) function of the independent variable x.
- y is a *solution* of (1) provided the equation holds for *all* x (in the domain specified).
- The highest derivative occurring in (1) is called the *order* of the equation.

Some familiar ODEs

You've probably seen the following examples in Calculus II.

1. The solutions of the ODE $y' = ky$ are $y = Ce^{kx}$, for an arbitrary constant C.
2. The solutions of the ODE $y'' - y = 0$ are $y = C_1e^t + C_2e^{-t}$, for arbitrary constants C_1 and C_2.
3. The solutions of the ODE $y'' + y = e^t$ are $y = C_1 \cos t + C_2 \sin t + \frac{1}{2}e^t$, for arbitrary constants C_1 and C_2.

You may want to go back and familiarize yourself with just how these solutions are found.

The definition of a Partial Differential Equation (PDE)

PDEs are the multivariable analogues of ODEs. As such, they involve *partial derivatives* of an unspecified function.

Specifically, a *Partial Differential Equation* (PDE) has the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, \underbrace{\ldots\ldots\ldots}_{\text{higher order partial derivatives of } u}) = 0 \qquad (2)$$

where:

- $u = u(x_1, x_2, \ldots, x_n)$ is an (unknown) function of the independent variables $x_1, x_2, \ldots, x_n$.
- u is a *solution* of (2) provided the equation holds for *all* $x_1, x_2, \ldots, x_n$ (in the domain specified).

Remarks:

Regarding the general PDE

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, \underbrace{\ldots\ldots\ldots}_{\text{higher order partial derivatives of } u}) = 0. \qquad (3)$$

1. Recall that $u_x = \frac{\partial u}{\partial x}$, $u_{xy} = \frac{\partial^2 u}{\partial x \, \partial y}$, etc. We will use these notations interchangeably.
2. Although every PDE can be put in the form (3), this is not always necessary.
3. When $n \leq 4$, we usually use more familiar independent variables, e.g. x, y, z, t.
4. The *order* of the PDE (3) is the highest (partial) derivative that explicitly occurs in the equation.

Examples

1. The function $u(x,y) = x^2 + y^2$ solves the (first order) PDE $xu_x + yu_y = 2u$.
2. The function $u(x,y) = e^{x-y}$ solves the (first order) PDE $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$.
3. The function $u(x,y) = x^2 - y^2$ solves the (second order) PDE $u_{xx} + u_{yy} = 0$.
4. The function $u(x,y) = (\sin x)(e^y + e^{-y})$ *also* solves $u_{xx} + u_{yy} = 0$.
5. The function $u(x,y,z) = xyz$ solves the (third order) PDE $u_{xyz} = 1$.

More remarks

1. As with ODEs, checking that a *given* function solves a PDE is straightforward.

2. The hard part is *finding* the solutions to a given PDE.

   - Solution spaces tend to be *infinite dimensional*. It's not usually possible to write down *every* solution.

   - There are no known techniques that will solve all PDEs.

More remarks

3. However, there are some very powerful techniques that are available for certain classes of PDEs:
    - Method of characteristics
    - Separation of variables, principle of superposition and Fourier series
    - Sturm-Liouville theory
4. One can also approximate solutions via numerical methods.
    - Often necessary for extremely complicated problems.
    - Usually studied in other courses, e.g. Heat Transfer.

Physical phenomena

Many physical phenomena can be effectively modeled via PDEs.

Before we can state them, recall that

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \cdots$$

is the *gradient operator* and

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \cdots$$

is the *Laplacian*.

Examples

1. **The wave equation:** If $u(\mathbf{x}, t)$ measures the displacement of an ideal elastic membrane from its equilibrium position, then u satisfies the (second order) PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u.$$

2. **The heat equation:** If $u(\mathbf{x}, t)$ gives the temperature in a perfectly thermally conductive medium, then u satisfies the (second order) PDE

$$\frac{\partial u}{\partial t} = c^2 \Delta u.$$

Examples

3. **The transport equation:** If $u(\mathbf{x}, t)$ is the concentration of a contaminant flowing though a fluid moving with velocity $\mathbf{v}$, then u satisfies the (first order) PDE

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = 0.$$

4. **The Laplace equation:** If $u(\mathbf{x})$ is the steady state temperature in a perfectly thermally conductive medium, then u satisfies the (second order) PDE

$$\Delta u = 0.$$

Examples

5. **The (1-D) KdV equation:** If $u(x, t)$ is the vertical displacement of a flowing shallow fluid, then u satisfies the (third order) PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

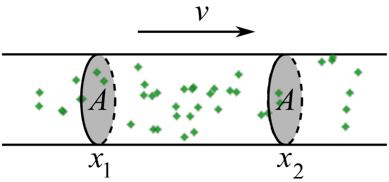
6. **The Schrödinger equation:** If $u(\mathbf{x}, t)$ is the wave-function of a quantum particle with mass μ, subject to a potential $V(\mathbf{x})$, then u satisfies the (second order) PDE

$$i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta u + V(\mathbf{x})u.$$

The 1-D transport equation

The set up:

- Consider a fluid flowing with velocity v though a capillary (of unbounded length) with cross-sectional area A.
- We introduce a "contaminant" to the fluid, and let $u(x, t)$ denote its concentration at position x and time t.



- At a fixed time t, the total amount of contaminant between positions x_1 and x_2 is

$$T_1(x_1, x_2, t) = \int_{x_1}^{x_2} u(x,t) \cdot A \, dx. \tag{4}$$

- Similarly, at a fixed position x, the total amount of contaminant that flows through from time t_1 to t_2 is

$$T_2(t_1, t_2, x) = \int_{t_1}^{t_2} u(x,t) \cdot A \cdot v \, dt. \tag{5}$$

We now "compute the same quantity in two different ways."

According to (4), the change in the amount of contaminant in the interval $[x_1, x_2]$ from time t_1 to t_2 is

$$
\begin{aligned}
T_1(x_1, x_2, t_2) - T_1(x_1, x_2, t_1) &= A \int_{x_1}^{x_2} u(x, t_2) - u(x, t_1)\, dx \\
&= A \int_{x_1}^{x_2} \int_{t_1}^{t_2} u_t(x, t)\, dt\, dx, \quad (6)
\end{aligned}
$$

where we have used the Fundamental Theorem of Calculus in the final line.

By the same token, (5) tells us the same quantity is also given by

$$\begin{aligned} T_2(t_1,t_2,x_1) - T_2(t_1,t_2,x_2) &= Av\int_{t_1}^{t_2} u(x_1,t) - u(x_2,t)\,dt \\ &= Av\int_{t_1}^{t_2}\int_{x_2}^{x_1} u_x(x,t)\,dx\,dt \\ &= -Av\int_{x_1}^{x_2}\int_{t_1}^{t_2} u_x(x,t)\,dt\,dx \quad (7) \end{aligned}$$

where we have used Fubini's theorem to reverse the order of integration (assuming the partial derivatives of the function $u(x,t)$ are sufficiently smooth).

Since we have simply computed the same quantity in two different ways, (6) and (7) are, in fact, the same:

$$A \int_{x_1}^{x_2} \int_{t_1}^{t_2} u_t(x,t)\,dt\,dx = -Av \int_{x_1}^{x_2} \int_{t_1}^{t_2} u_x(x,t)\,dt\,dx.$$

Moving everything to one side of the equation yields

$$A \int_{x_1}^{x_2} \int_{t_1}^{t_2} u_t(x,t) + v u_x(x,t)\,dt\,dx = 0.$$

The result

Since $A > 0$ and x_1, x_2, t_1, t_2 are arbitrary, this can only occur provided

$$u_t(x, t) + v u_x(x, t) = 0$$

for all (x, t).

Or, equivalently,

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0.$$

This is the **one-dimensional transport equation**.

Now what?

- Can we even produce *a single* solution to the transport equation?

$$\text{Yes: } u(x,t) = x - vt \text{ works.}$$

- Can we possibly find *every* solution? If so, by what means?

$$\text{Yes: } u(x,t) = f(x - vt), \text{ where } f \text{ is arbitrary.}$$

- We'll answer both of these next time!