

# Solving First Order PDEs

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Partial Differential Equations

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## Solving the transport equation

**Goal:** Determine every function  $u(x, t)$  that solves

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0,$$

where  $v$  is a fixed constant.

**Idea:** Perform a *linear change of variables* to eliminate one partial derivative:

$$\alpha = ax + bt,$$

$$\beta = cx + dt,$$

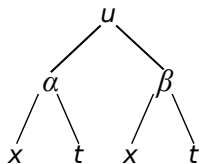
where:

$x, t$  : original independent variables,

$\alpha, \beta$  : new independent variables,

$a, b, c, d$  : constants to be chosen “conveniently,”  
must satisfy  $ad - bc \neq 0$ .

We use the *multivariable chain rule* to convert to  $\alpha$  and  $\beta$  derivatives:



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial x} = a \frac{\partial u}{\partial \alpha} + c \frac{\partial u}{\partial \beta},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial t} = b \frac{\partial u}{\partial \alpha} + d \frac{\partial u}{\partial \beta}.$$

Hence

$$\begin{aligned} \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} &= \left( b \frac{\partial u}{\partial \alpha} + d \frac{\partial u}{\partial \beta} \right) + v \left( a \frac{\partial u}{\partial \alpha} + c \frac{\partial u}{\partial \beta} \right) \\ &= (b + av) \frac{\partial u}{\partial \alpha} + (d + cv) \frac{\partial u}{\partial \beta}. \end{aligned}$$

Choosing  $a = 0$ ,  $b = 1$ ,  $c = 1$ ,  $d = -v$ , the original PDE becomes

$$\frac{\partial u}{\partial \alpha} = 0.$$

This tells us that

$$u = f(\beta) = f(cx + dt) = f(x - vt)$$

for *any* (differentiable) function  $f$ .

### Theorem

The general solution to the transport equation  $\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$  is given by

$$u(x, t) = f(x - vt),$$

where  $f$  is any differentiable function of one variable.

**Example**

Solve the transport equation  $\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$  given the initial condition

$$u(x, 0) = xe^{-x^2}, \quad -\infty < x < \infty.$$

**Solution:** We know that the general solution is given by

$$u(x, t) = f(x - 3t).$$

To find  $f$  we use the initial condition:

$$f(x) = f(x - 3 \cdot 0) = u(x, 0) = xe^{-x^2}.$$

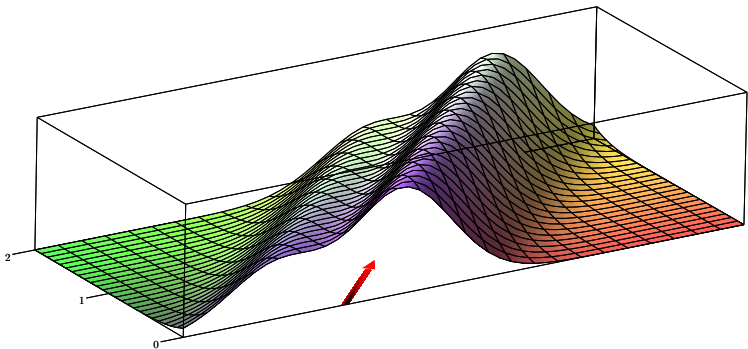
Thus

$$u(x, t) = (x - 3t)e^{-(x-3t)^2}.$$

# Interpreting the solutions of the transport equation

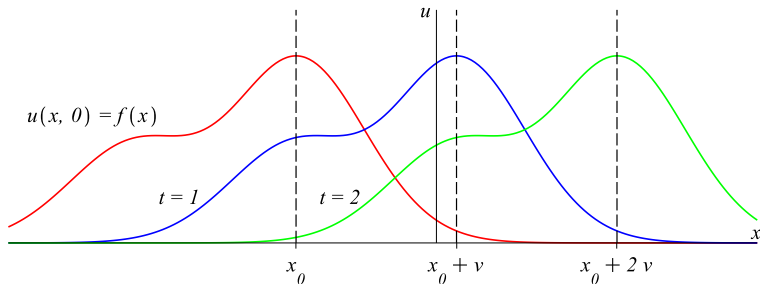
In three dimensions ( $xtu$ -space):

- The graph of the solution is the surface obtained by translating  $u = f(x)$  along the vector  $\mathbf{v} = \langle v, 1 \rangle$ ;
- The solution is constant along lines (in the  $xt$ -plane) parallel to  $\mathbf{v}$ .



If we plot the solution  $u(x, t) = f(x - vt)$  in the  $xu$ -plane, and animate  $t$ :

- $f(x) = u(x, 0)$  is the *initial condition* (concentration);
- $u(x, t)$  is a *traveling wave* with velocity  $v$  and shape given by  $u = f(x)$ .



**In general:** a linear change of variables can always be used to convert a PDE of the form

$$A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} = C(x, y, u)$$

into an “ODE,” i.e. a PDE containing only one partial derivative.

### Example

Solve  $5 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x$  given the initial condition

$$u(x, 0) = \sin 2\pi x, \quad -\infty < x < \infty.$$

**Solution:** As above, we perform the linear change of variables

$$\alpha = ax + bt,$$

$$\beta = cx + dt.$$



We find that

$$\begin{aligned}5\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= 5\left(b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}\right) + \left(a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}\right) \\ &= (a + 5b)\frac{\partial u}{\partial \alpha} + (c + 5d)\frac{\partial u}{\partial \beta}.\end{aligned}$$

We choose  $a = 1$ ,  $b = 0$ ,  $c = 5$ ,  $d = -1$ . Note that

$$\begin{aligned}ad - bc &= -1 \neq 0, \\ \alpha &= ax + bt = x, \\ \beta &= cx + dt = 5x - t.\end{aligned}$$

So the PDE (in the variables  $\alpha$ ,  $\beta$ ) becomes

$$\frac{\partial u}{\partial \alpha} = \alpha.$$

Integrating with respect to  $\alpha$  yields

$$u = \frac{\alpha^2}{2} + f(\beta) = \frac{x^2}{2} + f(5x - t).$$

The initial condition tells us that

$$\frac{x^2}{2} + f(5x) = u(x, 0) = \sin 2\pi x.$$

If we replace  $x$  with  $x/5$ , we get

$$f(x) = \sin \frac{2\pi x}{5} - \frac{x^2}{50}.$$

Therefore

$$\begin{aligned}u(x, t) &= \frac{x^2}{2} + f(5x - t) \\&= \frac{x^2}{2} + \sin \frac{2\pi(5x - t)}{5} - \frac{(5x - t)^2}{50} \\&= \frac{xt}{5} - \frac{t^2}{50} + \sin \frac{2\pi(5x - t)}{5}.\end{aligned}$$

**Remark:** There are an infinite number of choices for  $a, b, c, d$  that will “correctly” eliminate either  $\alpha$  or  $\beta$  from the PDE. Although they may appear different, the solutions obtained are always independent of the choice made.

# Characteristic curves

**Goal:** Develop a technique to solve the (somewhat more general) first order PDE

$$\frac{\partial u}{\partial x} + p(x, y) \frac{\partial u}{\partial y} = 0. \quad (1)$$

**Idea:** Look for *characteristic curves* in the  $xy$ -plane along which the solution  $u$  satisfies an ODE.

Consider  $u$  along a curve  $y = y(x)$ . On this curve we have

$$\frac{d}{dx} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}. \quad (2)$$

Comparing (1) and (2), if we require

$$\frac{dy}{dx} = p(x, y), \quad (3)$$

then the PDE becomes the ODE

$$\frac{d}{dx}u(x, y(x)) = 0. \quad (4)$$

These are the *characteristic ODEs* of the original PDE.

If we express the general solution to (3) in the form  $\varphi(x, y) = C$ , each value of  $C$  gives a characteristic curve.

Equation (4) says that  $u$  is constant along the characteristic curves, so that

$$u(x, y) = f(C) = f(\varphi(x, y)).$$

# The Method of Characteristics - Special Case

Summarizing the above we have:

## Theorem

*The general solution to*

$$\frac{\partial u}{\partial x} + p(x, y) \frac{\partial u}{\partial y} = 0$$

*is given by*

$$u(x, y) = f(\varphi(x, y)),$$

*where:*

- $\varphi(x, y) = C$  gives the general solution to  $\frac{dy}{dx} = p(x, y)$ , and
- $f$  is any differentiable function of one variable.

**Example**

Solve  $2y \frac{\partial u}{\partial x} + (3x^2 - 1) \frac{\partial u}{\partial y} = 0$  by the method of characteristics.

**Solution:** We first divide the PDE by  $2y$  obtaining

$$\frac{\partial u}{\partial x} + \underbrace{\frac{3x^2 - 1}{2y}}_{p(x,y)} \frac{\partial u}{\partial y} = 0.$$

So we need to solve

$$\frac{dy}{dx} = \frac{3x^2 - 1}{2y}.$$

This is separable:

$$2y \, dy = (3x^2 - 1) \, dx.$$

$$\int 2y \, dy = \int 3x^2 - 1 \, dx$$
$$y^2 = x^3 - x + C.$$

We can put this in the form  $y^2 - x^3 + x = C$  and hence

$$u(x, y) = f(y^2 - x^3 + x).$$

**Remark:** This technique can be generalized to PDEs of the form

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = C(x, y, u).$$



**Example**

$$\text{Solve } \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = u.$$

As above, along a curve  $y = y(x)$  we have

$$\frac{d}{dx} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}.$$

Comparison with the original PDE gives the characteristic ODEs

$$\begin{aligned} \frac{dy}{dx} &= x, \\ \frac{d}{dx} u(x, y(x)) &= u(x, y(x)). \end{aligned}$$

The first tells us that

$$y = \frac{x^2}{2} + y(0),$$

and the second that <sup>1</sup>

$$u(x, y(x)) = u(0, y(0))e^x = f(y(0))e^x.$$

Combining these gives

$$u(x, y) = f\left(y - \frac{x^2}{2}\right)e^x.$$

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<sup>1</sup>Recall that the solution to the ODE  $\frac{dw}{dx} = kw$  is  $w = Ce^{kx}$ . Since  $w(0) = Ce^0 = C$ , we can write this as  $w = w(0)e^{kx}$ .

# Summary

Consider a first order PDE of the form

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = C(x, y, u). \quad (5)$$

- When  $A(x, y)$  and  $B(x, y)$  are *constants*, a linear change of variables can be used to convert (5) into an “ODE.”
- In general, the method of characteristics yields a system of ODEs equivalent to (5).

In principle, these ODEs can always be solved completely to give the general solution to (5).