The One-Dimensional Wave Equation Revisited

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Partial Differential Equations
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The vibrating string ... again!

Recall: The motion of an ideal string of length L can be modeled by the 1-D wave equation

$$u_{tt} = c^2 u_{xx} \ \ (0 < x < L,\ t > 0),$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) = u(L,t) = 0 & \qquad (t > 0), \\ u(x,0) = f(x), & \\ u_t(x,0) = g(x) & \qquad (0 < x < L). \end{aligned}$$

Remarks:

- Previously: we attempted to express $u(x,t)$ as a series using the principle of superposition. This led to the need for Fourier series.
- Now: we will motivate and complete our earlier procedure.

Separation of variables

We seek "simple" solutions of the form

$$u(x,t) = X(x)T(t).$$

Differentiating yields

$$u_{tt} = XT'', \; u_{xx} = X''T.$$

Plugging into the wave equation gives $XT'' = c^2X''T$, or

$$\text{function of } x \text{ } \textit{only} \longrightarrow \frac{X''}{X} = \frac{T''}{c^2T} \longleftarrow \text{function of } t \text{ } \textit{only}$$

Since x and t are *independent*, both sides must be *constant*.

We introduce the *separation constant* k:

$$\frac{X''}{X} = k = \frac{T''}{c^2 T}.$$

This yields two ODEs in X and T:

$$X'' - kX = 0, \;\; T'' - kc^2 T = 0.$$

Imposing the boundary conditions we find that

$$\begin{aligned} 0 = u(0,t) = X(0)T(t) \;\Rightarrow\; X(0) = 0, \\ 0 = u(L,t) = X(L)T(t) \;\Rightarrow\; X(L) = 0. \end{aligned}$$

This gives us a *boundary value problem* in X:

$$X'' - kX = 0, \;\; X(0) = X(L) = 0. \tag{1}$$

Solving for X

We now determine the values of k for which (1) has nontrivial solutions.

Case 1: $k = \mu^2 > 0$. We need to solve $X'' - \mu^2 X = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r = \pm\mu,$$

which gives the general solution $X = c_1 e^{\mu x} + c_2 e^{-\mu x}$. The boundary conditions tell us that

$$c_1 + c_2 = c_1 e^{\mu L} + c_2 e^{-\mu L} = 0,$$

or in matrix form

$$\begin{pmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant here is $e^{-\mu L} - e^{\mu L} \neq 0$, which means that $c_1 = c_2 = 0$. So the only solution to the BVP in this case is $X \equiv 0$.

Case 2: $k = 0$. We need to solve $X'' = 0$. Integrating twice gives $X = c_1 x + c_2$.

The boundary conditions give $c_2 = c_1 L + c_2 = 0$, which imply that $c_1 = c_2 = 0$, and hence $X \equiv 0$ again.

Case 3: $k = -\mu^2 < 0$. We need to solve $X'' + \mu^2 X = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r = \pm i\mu,$$

which gives the general solution $X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions tell us that

$$c_1 = c_1 \cos(\mu L) + c_2 \sin(\mu L) = 0.$$

We will have *nontrivial* solutions iff $\sin(\mu L) = 0$. This happens iff $\mu L \in \pi\mathbb{Z}$, or

$$\mu = \mu_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}.$$

Choosing $c_2 = 1$ for convenience, we obtain the solutions

$$X = X_n = \sin(\mu_n x) = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

Remarks:

- We can omit $n \leq 0$ since they just yield multiples of these solutions.
- Up to the choice of the constant, these are the *only* nontrivial solutions to the BVP for X.

Solving for T

Having determined the X portion of our separated solution, we now turn to T.

Given any $n \in \mathbb{N}$, the separation constant in Case 3 is $k = -\mu_n^2$.

So T solves $T'' - kc^2 T = T'' + (\mu_n c)^2 T = 0$. The characteristic equation is

$$r^2 + (\mu_n c)^2 = 0 \quad \Rightarrow \quad r = \pm i \mu_n c,$$

which gives the general solution

$$T = T_n = b_n \cos(\mu_n ct) + b_n^* \sin(\mu_n ct) = b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t),$$

where:

- b_n and b_n^* are constants (to be determined later);
- $\lambda_n = \mu_n c = c\dfrac{n\pi}{L}$.

The normal modes

Putting the two factors together we obtain the *normal modes* of the wave equation (for $n \in \mathbb{N}$)

$$u_n(x,t) = X_n(x)\,T_n(t) = \sin(\mu_n x)\left(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)\right).$$

Remarks:

- The nth normal mode:
  - $*$ is spatially $2\pi/\mu_n = 2L/n$-periodic;
  - $*$ is temporally $2\pi/\lambda_n = 2L/nc$-periodic.
- As n increases, the normal modes oscillate more rapidly (in space *and* time).
- Up to a scalar multiple and a phase shift (in time) the modes are all of the form $\sin(\mu_n x)\cos(\lambda_n t)$.

Superposition

Recall: Because the functions u_n solve the vibrating string problem, the *principle of superposition* ensures that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin(\mu_n x)\left(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)\right)$$

solves it, too.

Remarks:

- Because it is a common period for each summand, we see that $2L/c$ is a temporal period for this solution.
- Although this solves the wave equation and has fixed endpoints, we have yet to impose the initial conditions.

Initial conditions

We now use the initial conditions to determine $\{b_n\}$ and $\{b_n^*\}$.

Setting $t = 0$ yields

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\mu_n x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

which is the $2L$-periodic sine expansion of $f(x)$. Hence

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Now differentiate with respect to t and set $t = 0$:

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \lambda_n b_n^* \sin(\mu_n x) = \sum_{n=1}^{\infty} \lambda_n b_n^* \sin\left(\frac{n\pi x}{L}\right).$$

This is the $2L$-periodic sine expansion of $g(x)$. Hence

$$\lambda_n b_n^* = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,$$

or, since $\lambda_n = n\pi c/L$:

$$b_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Theorem (Series solution to the vibrating string problem)

The solution of the boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && (0 < x < L,\ t > 0),\\ u(0,t) &= u(L,t) = 0 && (t > 0),\\ u(x,0) &= f(x),\ \ u_t(x,0) = g(x) && (0 < x < L) \end{aligned}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\mu_n x)\left(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)\right)$$

where $\mu_n = \frac{n\pi}{L}$, $\lambda_n = \mu_n c$ *and*

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,\ \ b_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Remarks

- Note that the initial shape and velocity influence the solution *independently*. In particular:
  - $*$ If $f(x) \equiv 0$, then $b_n = 0$ for all n.
  - $*$ If $g(x) \equiv 0$, then $b_n^* = 0$ for all n.
- The solution can also be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\mu_n x)\cos(\lambda_n t) + \sum_{n=1}^{\infty} b_n^* \sin(\mu_n x)\sin(\lambda_n t).$$

- Note that

$$b_n = (n\text{th } 2L\text{-periodic sine series coeff. of } f),$$

$$b_n^* = \frac{1}{\lambda_n}(n\text{th } 2L\text{-periodic sine series coeff. of } g).$$

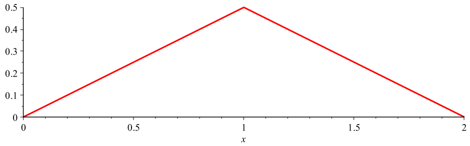
  So, if the sine series of f or g are known, we need not use the integral formulae.

Example

Solve the vibrating string problem

$$
\begin{aligned}
&u_{tt} = 100u_{xx} && (0 < x < 2,\ t > 0),\\
&u(0,t) = u(2,t) = 0 && (t > 0),\\
&u(x,0) = \begin{cases} \frac{x}{2} & \textit{if } 0 \le x < 1,\\ 1 - \frac{x}{2} & \textit{if } 1 \le x \le 2, \end{cases}\\
&u_t(x,0) = 0.
\end{aligned}
$$

We have $L = 2$, $c = 10$ and $b_n^* = 0$ for all n. Here's the initial shape ($f(x)$):



According to exercise 2.4.17b (with $p = L = 2$, $a = 1$ and $h = 1/2$):

$$f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} \sin\left(\frac{n\pi x}{2}\right) \quad \Rightarrow \quad b_n = \frac{4\sin(n\pi/2)}{\pi^2 n^2}.$$

We therefore have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\mu_n x) \cos(\lambda_n t) \\ &= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} \sin\left(\frac{n\pi x}{2}\right) \cos(5n\pi t), \qquad \text{(A)} \end{aligned}$$

since $\mu_n = n\pi/2$ and $\lambda_n = \mu_n c = 5n\pi$.

Example

Suppose that in the preceding problem we instead require that $u_t(x, 0) = 1$ for $0 < x < 2$. Find $u(x, t)$ in this case.

We only need to find b_n^* and add to our earlier work.

By exercise 2.3.1, the 4-periodic sine series for $g(x) = 1$ is

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Note *only odd indexed modes occur*. Therefore

$$\lambda_{2k+1} b_{2k+1}^* = \frac{4}{(2k+1)\pi}$$

$$\Rightarrow \quad b_{2k+1}^* = \frac{4}{\lambda_{2k+1}(2k+1)\pi} = \frac{4}{5(2k+1)^2\pi^2}.$$

It follows that the b_n^* portion of the solution is

$$\sum_{n=1}^{\infty} b_n^* \sin(\mu_n x)\sin(\lambda_n t) = \sum_{k=0}^{\infty} b_{2k+1}^* \sin(\mu_{2k+1} x)\sin(\lambda_{2k+1} t)$$

$$= \frac{4}{5\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sin\left(\frac{(2k+1)\pi x}{2}\right)\sin(5(2k+1)\pi t)\,. \quad \text{(B)}$$

The overall solution is the sum of this and our previous answer:

$$u(x,t) = \text{(A)} + \text{(B)}.$$