The One-Dimensional Heat Equation: Neumann and Robin boundary conditions

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Partial Differential Equations
February 26, 2015

Solving the Heat Equation

Case 4: inhomogeneous Neumann boundary conditions

Continuing our previous study, let's now consider the heat problem

$$
\begin{aligned}
&u_t = c^2 u_{xx} && (0 < x < L\ ,\ 0 < t),\\
&u_x(0,t) = -F_1,\ \ u_x(L,t) = -F_2 && (0 < t),\\
&u(x,0) = f(x) && (0 < x < L).
\end{aligned}
$$

This models the temperature in a wire of length L with given initial temperature distribution and *constant heat flux* at each end.

Remark: In fact, according to Fourier's law of heat conduction

$$
\begin{aligned}
\text{heat flux } \textit{in} \text{ at left end } &= K_0 F_1,\\
\text{heat flux } \textit{out} \text{ at right end } &= K_0 F_2,
\end{aligned}
$$

where K_0 is the wire's thermal conductivity.

Homogenizing the boundary conditions

As in the case of inhomogeneous Dirichlet conditions, we reduce to a homogenous problem by subtracting a "special" function. Let

$$u_1(x,t) = \frac{F_1 - F_2}{2L}x^2 - F_1 x + \frac{c^2(F_1 - F_2)}{L}t.$$

One can easily show that u_1 solves the heat equation and

$$\frac{\partial u_1}{\partial x}(0,t) = -F_1 \quad \text{and} \quad \frac{\partial u_1}{\partial x}(L,t) = -F_2.$$

By superposition, $u_2 = u - u_1$ solves the "homogenized" problem

$$\begin{aligned}
&u_t = c^2 u_{xx} && (0 < x < L\,,\ 0 < t),\\
&u_x(0,t) = u_x(L,t) = 0 && (0 < t),\\
&u(x,0) = f(x) - u_1(x,0) && (0 < x < L).
\end{aligned}$$

Complete solution

We therefore have the (analogous) solution procedure:

Step 1. Construct the special function u_1.

Step 2. Subtract u_1 from the original problem to "homogenize" it.

Step 3. Solve the "homogenized" problem for u_2.

Step 4. Construct the solution $u = u_1 + u_2$ to the original problem.

Remarks:

- According to earlier work, $\lim_{t\to\infty} u_2(x,t) = a_0$. So for large t:

$$u(x,t) \approx a_0 + u_1(x,t).$$

- The function $u_1(x,t)$ is *not* a steady state unless $F_1 = F_2$. Why? What does this mean physically?

Example

Solve the following heat problem:

$$\begin{aligned} &u_t = \frac{1}{4} u_{xx}, && 0 < x < 1 \ , \ 0 < t, \\ &u_x(0,t) = -5, \ \ u_x(1,t) = -2, && 0 < t, \\ &u(x,0) = 0, && 0 < x < 1. \end{aligned}$$

Since $c^2 = 1/4$, $F_1 = 5$ and $F_2 = 2$, the "homogenizing" function is

$$u_1(x,t) = \frac{3}{2}x^2 - 5x + \frac{3}{4}t.$$

Subtracting this from u yields a problem with homogeneous boundary conditions and initial condition

$$u(x,0) = 0 - u_1(x,0) = -\frac{3}{2}x^2 + 5x.$$

The solution of the "homogenized" problem is (HW)

$$u_2(x,t) = 2 + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{4(-1)^n - 10}{n^2}e^{-n^2\pi^2 t/4}\cos(n\pi x),$$

so that the solution of the original problem is

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) \\ &= \frac{3}{2}x^2 - 5x + \frac{3}{4}t + 2 + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{4(-1)^n - 10}{n^2}e^{-n^2\pi^2 t/4}\cos(n\pi x). \end{aligned}$$

Remark: As we mentioned above, this shows that for large t

$$u(x,t) \approx \frac{3}{2}x^2 - 5x + \frac{3}{4}t + 2.$$

Solving the Heat Equation

Case 5: mixed (Dirichlet and Robin) homogeneous boundary conditions

As a final case study, we now will solve the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L,\ 0 < t), \\ u(0,t) &= 0 && (0 < t), \\ u_x(L,t) &= -\kappa u(L,t) && (0 < t), \qquad (1) \\ u(x,0) &= f(x) && (0 < x < L). \end{aligned}$$

Remarks:

- The condition (1) is linear and homogeneous:

$$\kappa u(L,t) + u_x(L,t) = 0$$

  Recall that this is called a *Robin condition*.
- We take $\kappa > 0$. This means that the heat flux at the right end is proportional to the current temperature there.

Separation of variables

As before, the assumption that $u(x,t) = X(x)T(t)$ leads to the ODEs

$$X'' - kX = 0, \quad T' - c^2 kT = 0,$$

and the boundary conditions imply

$$X(0) = 0, \quad X'(L) = -\kappa X(L).$$

Case 1: $k = 0$. As usual, solving $X'' = 0$ gives $X = c_1 x + c_2$. The boundary conditions become

$$\begin{aligned} 0 = X(0) = c_2, \quad c_1 = X'(L) = -\kappa X(L) = -\kappa(c_1 L + c_2) \\ \Rightarrow \ c_1(1 + \kappa L) = 0 \ \Rightarrow \ c_1 = 0. \end{aligned}$$

Hence, $X \equiv 0$ in this case.

Case 2: $k = \mu^2 > 0$. Again we have $X'' - \mu^2 X = 0$ and

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The boundary conditions become

$$0 = c_1 + c_2, \quad \mu(c_1 e^{\mu L} - c_2 e^{-\mu L}) = -\kappa(c_1 e^{\mu L} + c_2 e^{-\mu L}),$$

or in matrix form

$$\begin{pmatrix} 1 & 1 \\ (\kappa + \mu)e^{\mu L} & (\kappa - \mu)e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant is

$$(\kappa-\mu)e^{-\mu L} - (\kappa+\mu)e^{\mu L} = -\Big(\kappa(e^{\mu L} - e^{-\mu L}) + \mu(e^{\mu L} + e^{-\mu L})\Big) < 0,$$

so that $c_1 = c_2 = 0$ and $X \equiv 0$.

Case 3: $k = -\mu^2 < 0$. From $X'' + \mu^2 X = 0$ we find

$$X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

and from the boundary conditions we have

$$0 = c_1, \quad \mu(-c_1 \sin(\mu L) + c_2 \cos(\mu L)) = -\kappa(c_1 \cos(\mu L) + c_2 \sin(\mu L))$$
$$\Rightarrow \quad c_2 \left(\mu \cos(\mu L) + \kappa \sin(\mu L)\right) = 0.$$

So that $X \not\equiv 0$, we must have

$$\mu \cos(\mu L) + \kappa \sin(\mu L) = 0 \quad \Rightarrow \quad \tan(\mu L) = -\frac{\mu}{\kappa}.$$

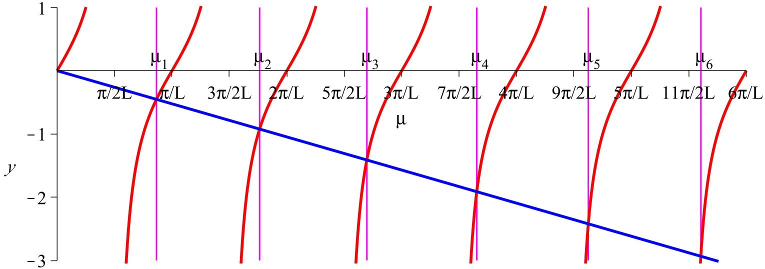
This equation has an infinite sequence of positive solutions

$$0 < \mu_1 < \mu_2 < \mu_3 < \cdots$$

and we obtain $X = X_n = \sin(\mu_n x)$ for $n \in \mathbb{N}$.

The solutions of $\tan(\mu L) = -\mu/\kappa$

The figure below shows the curves $y = \tan(\mu L)$ (in red) and $y = -\mu/\kappa$ (in blue).



The μ-coordinates of their intersections (in pink) are the values μ_1, μ_2, μ_3, ...

Remarks: From the diagram we see that:

- For each n, $(2n-1)\pi/2L < \mu_n < n\pi/L$.
- As $n \to \infty$, $\mu_n \to (2n-1)\pi/2L$.
- Smaller values of κ and L tend to accelerate this convergence.

Normal modes: As in the earlier situations, for each $n \in \mathbb{N}$ we have the corresponding

$$T = T_n = c_n e^{-\lambda_n^2 t},\ \lambda_n = c\mu_n$$

which gives the *normal mode*

$$u_n(x,t) = X_n(x)T_n(t) = c_n e^{-\lambda_n^2 t} \sin(\mu_n x).$$

Superposition

Superposition of normal modes gives the general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t} \sin(\mu_n x).$$

Imposing the initial condition gives us

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x).$$

This is a *generalized Fourier sine series* for $f(x)$. It is *different* from the ordinary sine series for $f(x)$ since

μ_n is not a multiple of π/L.

Generalized Fourier coefficients

To compute the *generalized Fourier coefficients* c_n we will use:

Theorem

The functions

$$X_1(x) = \sin(\mu_1 x), X_2(x) = \sin(\mu_2 x), X_3(x) = \sin(\mu_3 x), \ldots$$

form a complete orthogonal set on $[0, L]$.

- *Complete* means that all "sufficiently nice" functions can be represented via generalized Fourier series.
- Recall that the inner product of $f(x)$ and $g(x)$ on $[0, L]$ is

$$\langle f, g \rangle = \int_0^L f(x)g(x)\,dx.$$

"Extracting" the generalized Fourier coefficients

If

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x) = \sum_{n=1}^{\infty} c_n X_n(x),$$

the "usual" argument using the orthogonality of $\{X_1, X_2, X_3, \ldots\}$ on $[0, L]$ yields

$$c_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\displaystyle\int_0^L f(x) \sin(\mu_n x)\, dx}{\displaystyle\int_0^L \sin^2(\mu_n x)\, dx}$$

$$= \frac{2\kappa}{\kappa L + \cos^2(\mu_n L)} \int_0^L f(x) \sin(\mu_n x)\, dx,$$

the final step being left as an exercise.

Conclusion

Theorem

The solution to the heat problem with boundary and initial conditions

$$\begin{aligned} u(0,t) = 0, \;\; u_x(L,t) = -\kappa u(L,t) \qquad & (0 < t), \\ u(x,0) = f(x) \qquad & (0 < x < L) \end{aligned}$$

is given by $u(x,t) = \displaystyle\sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t} \sin(\mu_n x)$*, where* μ_n *is the nth positive solution to* $\tan(\mu L) = \dfrac{-\mu}{\kappa}$*,* $\lambda_n = c\mu_n$*, and*

$$c_n = \frac{\int_0^L f(x)\sin(\mu_n x)\,dx}{\int_0^L \sin^2(\mu_n x)\,dx} = \frac{2\kappa}{\kappa L + \cos^2(\mu_n L)} \int_0^L f(x)\sin(\mu_n x)\,dx.$$

Remarks:

- For any given $f(x)$ these integrals can be computed explicitly in terms of μ_n.
- The values of μ_n, however, must typically be found via numerical methods.

Example

Solve the following heat problem:

$$\begin{aligned} u_t &= \frac{1}{25}u_{xx} && (0 < x < 3,\ \ 0 < t),\\ u(0,t) &= 0,\ \ u_x(3,t) = -\frac{1}{2}u(3,t) && (0 < t),\\ u(x,0) &= 100\left(1 - \frac{x}{3}\right) && (0 < x < 3). \end{aligned}$$

We have $c = 1/5$, $L = 3$, $\kappa = 1/2$ and $f(x) = 100(1 - x/3)$.

The Fourier coefficients are given by

$$\begin{aligned} c_n &= \frac{1}{3/2+\cos^2(3\mu_n)} \int_0^3 100\left(1-\frac{x}{3}\right)\sin(\mu_n x)\,dx \\ &= \left(\frac{1}{3/2+\cos^2(3\mu_n)}\right)\left(\frac{100(3\mu_n-\sin(3\mu_n))}{3\mu_n^2}\right) \\ &= \frac{200(3\mu_n-\sin(3\mu_n))}{3\mu_n^2\left(3+2\cos^2(3\mu_n)\right)}. \end{aligned}$$

We therefore have

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(3\mu_n-\sin(3\mu_n))}{3\mu_n^2\left(3+2\cos^2(3\mu_n)\right)} e^{-\mu_n^2 t/25}\sin(\mu_n x),$$

where μ_n is the nth positive solution to $\tan(3\mu) = -2\mu$.

Remarks:

- In order to use this solution for numerical approximation or visualization, we must compute the values μ_n.
- This can be done numerically in Maple, using the `fsolve` command. Specifically, μ_n can be computed via the input

  ```
  fsolve(tan(m*L)=-m/k,m=(2*n-1)*Pi/(2*L)..n*Pi/L);
  ```

  where `L` and `k` have been assigned the values of L and κ, respectively.
- These values can be computed and stored in an `Array` structure, or one can define μ_n as a function using the `->` operator.

Here are approximations to the first 5 values of μ_n and c_n in the preceding example.

n	μ_n	c_n
1	0.7249	47.0449
2	1.6679	45.1413
3	2.6795	21.3586
4	3.7098	19.3403
5	4.7474	12.9674

Therefore

$$\begin{aligned} u(x,t) &= 47.0449e^{-0.0210t}\sin(0.7249x) + 45.1413e^{-0.1113t}\sin(1.6679x) \\ &\quad + 21.3586e^{-0.2872t}\sin(2.6795x) + 19.3403e^{-0.5505t}\sin(3.7098x) \\ &\quad + 12.9674e^{-0.9015t}\sin(4.7474x) + \cdots \end{aligned}$$