The Laplacian in Polar Coordinates

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Partial Differential Equations
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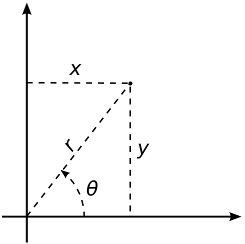
Polar coordinates

To solve boundary value problems on circular regions, it is convenient to switch from rectangular (x, y) to polar (r, θ) spatial coordinates:



$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$x^2 + y^2 = r^2.$$

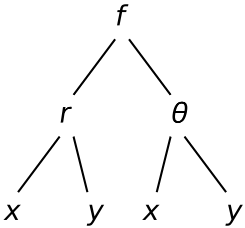
This requires us to express the rectangular Laplacian

$$\Delta u = u_{xx} + u_{yy}$$

in terms of derivatives with respect to r and θ.

The chain rule

For any function $f(r,\theta)$, we have the familiar tree diagram and chain rule formulae:



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

or

$$f_x = f_r r_x + f_\theta \theta_x$$

$$f_y = f_r r_y + f_\theta \theta_y$$

First take $f = u$ to obtain

$$u_x = u_r r_x + u_\theta \theta_x \quad \Rightarrow \quad u_{xx} = u_r r_{xx} + (u_r)_x r_x + u_\theta \theta_{xx} + (u_\theta)_x \theta_x.$$

Applying the chain rule with $f = u_r$ and then with $f = u_\theta$ yields

$$\begin{aligned} u_{xx} &= u_r r_{xx} + \left(u_{rr} r_x + u_{r\theta}\theta_x\right) r_x + u_\theta \theta_{xx} + \left(u_{\theta r} r_x + u_{\theta\theta}\theta_x\right)\theta_x \\ &= u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta}\theta_x^2. \end{aligned}$$

An entirely similar computation using y instead of x also gives

$$u_{yy} = u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta}\theta_y^2.$$

If we add these expressions and collect like terms we get

$$\begin{aligned} \Delta u = u_r\left(r_{xx} + r_{yy}\right) + u_{rr}\left(r_x^2 + r_y^2\right) + 2u_{r\theta}\left(r_x\theta_x + r_y\theta_y\right) \\ + u_\theta\left(\theta_{xx} + \theta_{yy}\right) + u_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right). \end{aligned}$$

Differentiate $x^2 + y^2 = r^2$ with respect to x and then y:

$$2x = 2rr_x \;\Rightarrow\; r_x = \frac{x}{r} \;\Rightarrow\; r_{xx} = \frac{r - xr_x}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3},$$

$$2y = 2rr_y \;\Rightarrow\; r_y = \frac{y}{r} \;\Rightarrow\; r_{yy} = \frac{r - yr_y}{r^2} = \frac{r^2 - y^2}{r^3} = \frac{x^2}{r^3}.$$

Now differentiate $\tan\theta = \frac{y}{x}$ with respect to x and then y:

$$\sec^2\theta\,\theta_x = -\frac{y}{x^2} \;\Rightarrow\; \theta_x = -\frac{y\cos^2\theta}{x^2} = -\frac{y}{r^2} \;\Rightarrow\; \theta_{xx} = \frac{2y}{r^3}r_x = \frac{2xy}{r^4},$$

$$\sec^2\theta\,\theta_y = \frac{1}{x} \;\Rightarrow\; \theta_y = \frac{\cos^2\theta}{x} = \frac{x}{r^2} \;\Rightarrow\; \theta_{yy} = \frac{-2x}{r^3}r_y = -\frac{2xy}{r^4}.$$

Together these yield

$$r_{xx}+r_{yy}=\frac{y^2+x^2}{r^3}=\frac{1}{r},\quad r_x^2+r_y^2=\frac{x^2+y^2}{r^2}=1.$$

$$\theta_{xx}+\theta_{yy}=\frac{2xy}{r^4}+\frac{-2xy}{r^4}=0,\quad \theta_x^2+\theta_y^2=\frac{y^2+x^2}{r^4}=\frac{1}{r^2},$$

$$r_x\theta_x+r_y\theta_y=\frac{-xy}{r^3}+\frac{yx}{r^3}=0,$$

and we finally obtain

$$\begin{aligned}\Delta u = & u_r\left(r_{xx}+r_{yy}\right)+u_{rr}\left(r_x^2+r_y^2\right)+2u_{r\theta}\left(r_x\theta_x+r_y\theta_y\right)\\ & +u_\theta\left(\theta_{xx}+\theta_{yy}\right)+u_{\theta\theta}\left(\theta_x^2+\theta_y^2\right)\\ = & \frac{1}{r}u_r+u_{rr}+\frac{1}{r^2}u_{\theta\theta}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}.\end{aligned}$$

Example

Use polar coordinates to show that the function $u(x, y) = \dfrac{y}{x^2 + y^2}$ *is harmonic.*

We need to show that $\Delta u = 0$. In polar coordinates we have

$$u(r, \theta) = \frac{r \sin\theta}{r^2} = \frac{\sin\theta}{r}$$

so that

$$u_r = -\frac{\sin\theta}{r^2}, \qquad u_{rr} = \frac{2\sin\theta}{r^3}, \qquad u_{\theta\theta} = \frac{-\sin\theta}{r},$$

and thus

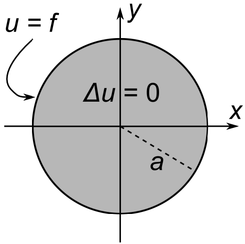
$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{2\sin\theta}{r^3} - \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} = 0.$$

The Dirichlet problem on a disk

Goal: Solve the Dirichlet problem on a disk of radius a, centered at the origin. In polar coordinates this has the form

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r < a,$$

$$u(a,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi.$$



Remarks:

- We will require that f is 2π-periodic.
- Likewise, we require that $u(r,\theta)$ is 2π-periodic in θ.

Separation of variables

If we assume that $u(r,\theta) = R(r)\Theta(\theta)$ and plug into $\Delta u = 0$, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \;\Rightarrow\; r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

$$\Rightarrow\; r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

This yields the pair of separated ODEs

$$r^2R'' + rR' - \lambda R = 0 \quad \text{and} \quad \Theta'' + \lambda\Theta = 0.$$

We also have the "boundary conditions"

$$\Theta \text{ is } 2\pi\text{-periodic} \qquad \text{and} \qquad R(0+) \text{ is finite.}$$

Solving for Θ

The solutions of $\Theta'' + \lambda\Theta = 0$ are periodic only if

$$\lambda = \mu^2 \geq 0 \quad \Rightarrow \quad \Theta = a\cos(\mu\theta) + b\sin(\mu\theta).$$

In order for the period to be 2π we also need

$$1 = \cos(0\mu) = \cos(2\pi\mu) \quad \Rightarrow \quad 2\pi\mu = 2\pi n \quad \Rightarrow \quad \mu = n \in \mathbb{N}_0.$$

Hence $\lambda = n^2$ and

$$\Theta = \Theta_n = a_n\cos(n\theta) + b_n\sin(n\theta), \quad n \in \mathbb{N}_0.$$

It follows that R satisfies

$$r^2R'' + rR' - n^2R = 0,$$

which is called an *Euler equation*.

Interlude
Euler equations

An *Euler equation* is a second order ODE of the form

$$x^2y'' + \alpha xy' + \beta y = 0.$$

Its solutions are determined by the roots of its *indicial equation*

$$\rho^2 + (\alpha - 1)\rho + \beta = 0.$$

Case 1: If the roots are $\rho_1 \neq \rho_2$, then the general solution is

$$y = c_1 x^{\rho_1} + c_2 x^{\rho_2}.$$

Case 2: If there is only one root ρ_1, then the general solution is

$$y = c_1 x^{\rho_1} + c_2 x^{\rho_1} \ln x.$$

Solving for R

The indicial equation of $r^2R'' + rR' - n^2R = 0$ is

$$\rho^2 + (1-1)\rho - n^2 = \rho^2 - n^2 = 0 \;\Rightarrow\; \rho = \pm n.$$

This means that

$$\begin{aligned} R &= c_1 r^n + c_2 r^{-n} \quad (n \neq 0), \\ R &= c_1 + c_2 \ln r \qquad (n = 0). \end{aligned}$$

These will be finite at $r = 0$ only if $c_2 = 0$. Setting $c_1 = a^{-n}$ gives

$$R = R_n = \left(\frac{r}{a}\right)^n, \quad n \in \mathbb{N}_0.$$

Separated solutions and superposition

We therefore obtain the separated solutions

$$u_n(r,\theta) = R_n(r)\Theta_n(\theta) = \left(\frac{r}{a}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta)), \quad n \in \mathbb{N}_0.$$

Noting that

$$u_0(r,\theta) = \left(\frac{r}{a}\right)^0 (a_0\cos 0 + b_0\sin 0) = a_0,$$

superposition gives the general solution

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta)).$$

Boundary values and conclusion

Imposing our Dirichlet boundary conditions gives

$$f(\theta) = u(a,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

which is just the ordinary 2π-periodic Fourier series for f!

Theorem

The solution of the Dirichlet problem on the disk of radius a centered at the origin, with boundary condition $u(a,\theta) = f(\theta)$ is $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$, where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta,$$

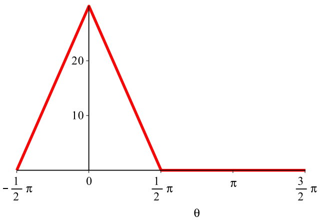
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta.$$

Example

Find the solution to the Dirichlet problem on a disk of radius 3 with boundary values given by

$$f(\theta) = \begin{cases} \frac{30}{\pi}(\pi + 2\theta) & \text{if } \frac{-\pi}{2} \le \theta < 0, \\ \frac{30}{\pi}(\pi - 2\theta) & \text{if } 0 \le \theta < \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} \le \theta < \frac{3\pi}{2}. \end{cases}$$

We have $a = 3$. The graph of f is

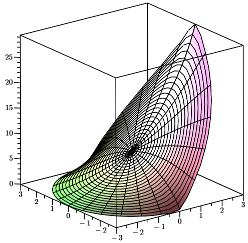


According to exercise 2.3.8 (with $p = \pi$, $c = 30$ and $d = \pi/2$):

$$f(\theta) = \frac{15}{2} + \frac{120}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi/2)}{n^2} \cos(n\theta).$$

Hence, the solution to the Dirichlet problem is

$$u(r,\theta) = \frac{15}{2} + \frac{120}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{r}{3}\right)^n \frac{1 - \cos(n\pi/2)}{n^2} \cos(n\theta).$$



Example

Solve the Dirichlet problem on a disk of radius 2 with boundary values given by $f(\theta) = \cos^2\theta$. Express your answer in cartesian coordinates.

We have $a = 2$ and

$$f(\theta) = \cos^2\theta = \frac{1+\cos(2\theta)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2\theta),$$

which is a finite 2π-periodic Fourier series (i.e. $a_0 = 1/2$, $a_2 = 1/2$, and all other coefficients are zero).
Hence

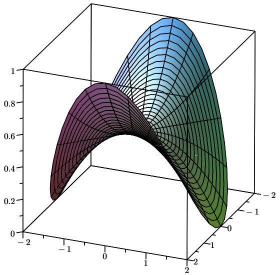
$$u(r,\theta) = \frac{1}{2} + \left(\frac{r}{2}\right)^2 \cdot \frac{1}{2}\cos(2\theta) = \frac{1}{2} + \frac{r^2\cos(2\theta)}{8}.$$

Since $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, we find that

$$r^2\cos(2\theta) = r^2\cos^2\theta - r^2\sin^2\theta = x^2 - y^2$$

and hence

$$u = \frac{1}{2} + \frac{r^2\cos(2\theta)}{8} = \frac{1}{2} + \frac{x^2 - y^2}{8}.$$



Example

Solve the Dirichlet problem on a disk of radius 1 if the boundary value is 50 in the first quadrant, and zero elsewhere.

We are given $a = 1$, $f(\theta) = 50$ for $0 \leq \theta \leq \pi/2$ and $f(\theta) = 0$ otherwise. The Fourier coefficients of f are

$$a_0 = \frac{1}{2\pi}\int_0^{\pi/2} 50\, d\theta = \frac{25}{2},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi/2} 50\cos(n\theta)\, d\theta = \frac{50\sin(n\pi/2)}{n\pi},$$

$$b_n = \frac{1}{\pi}\int_0^{\pi/2} 50\sin(n\theta)\, d\theta = \frac{50(1-\cos(n\pi/2))}{n\pi},$$

so that

$$u(r,\theta) = \frac{25}{2} + \frac{50}{\pi}\sum_{n=1}^{\infty} r^n \left(\frac{\sin(n\pi/2)}{n}\cos(n\theta) + \frac{(1-\cos(n\pi/2))}{n}\sin(n\theta) \right).$$

Remarks:

- One can frequently use identities like (valid for $|r| < 1$)

$$\sum_{n=1}^{\infty} \frac{r^n \cos(n\theta)}{n} = -\frac{1}{2} \ln\left(1 - 2r\cos\theta + r^2\right),$$
$$\sum_{n=1}^{\infty} \frac{r^n \sin(n\theta)}{n} = \arctan\left(\frac{r\sin\theta}{1 - r\cos\theta}\right),$$

  to convert series solutions in polar coordinates to cartesian expressions.

- Using the second identity, one can show that the solution in the preceding example is

$$u(x,y) = \frac{25}{2} + \frac{50}{\pi}\left(\arctan\left(\frac{y}{1-x}\right) + \arctan\left(\frac{x}{1-y}\right)\right).$$