The two dimensional wave equation

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Partial Differential Equations
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Vibrating membranes

Goal: Model the motion of an ideal elastic membrane.

Set up: Assume the membrane at rest is a region of the xy-plane and let

$$u(x,y,t) = \begin{cases} \text{vertical deflection of membrane from equilib-} \\ \text{rium at position } (x,y) \text{ and time } t. \end{cases}$$

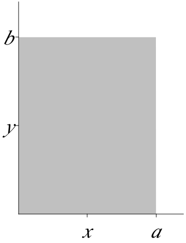
For a fixed t, the surface $z = u(x,y,t)$ gives the shape of the membrane at time t.

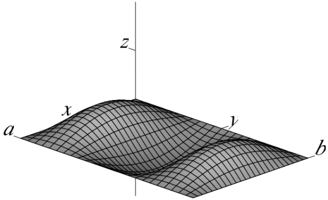
Under ideal assumptions (e.g. uniform membrane density, uniform tension, no resistance to motion, small deflection, etc.) one can show that u satisfies the *two dimensional wave equation*

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}).$$

Rectangular membranes

We assume the membrane lies over the rectangular region $R = [0, a] \times [0, b]$ and has fixed edges.





These facts are expressed by the *boundary conditions*

$$u(0, y, t) = u(a, y, t) = 0, \qquad 0 \le y \le b, \ t > 0,$$
$$u(x, 0, t) = u(x, b, t) = 0, \qquad 0 \le x \le a, \ t > 0.$$

We must also specify how the membrane is initially deformed and set into motion. This is done via the *initial conditions*

$$u(x,y,0) = f(x,y), \qquad (x,y) \in R,$$
$$u_t(x,y,0) = g(x,y), \qquad (x,y) \in R.$$

New goal: solve the 2-D wave equation subject to the boundary and initial conditions just given.

As usual, we will:

- Use separation of variables to find simple solutions satisfying the homogeneous boundary conditions; and
- Use the principle of superposition to build up a series solution that satisfies the initial conditions as well.

Separation of variables

We seek nontrivial solutions of the form

$$u(x,y,t) = X(x)Y(y)T(t).$$

Plugging this into $u_{tt} = c^2(u_{xx} + u_{yy})$ we get

$$XYT'' = c^2\left(X''YT + XY''T\right) \;\Rightarrow\; \frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Because the two sides are functions of different independent variables, they must be constant:

$$\frac{T''}{c^2T} = A = \frac{X''}{X} + \frac{Y''}{Y} \;\Rightarrow\; \begin{cases} T'' - c^2AT = 0, \\ \dfrac{X''}{X} = -\dfrac{Y''}{Y} + A. \end{cases}$$

Since the two sides again involve unrelated variables, both are constant:

$$\frac{X''}{X} = B = -\frac{Y''}{Y} + A.$$

Setting $C = A - B$, these equations can be rewritten as

$$X'' - BX = 0, \quad Y'' - CY = 0.$$

The first boundary condition is

$$0 = u(0, y, t) = X(0)Y(y)T(t).$$

Canceling Y and T yields $X(0) = 0$. Likewise, we obtain

$$X(a) = 0, \quad Y(0) = Y(b) = 0.$$

There are no boundary conditions on T.

We have already solved the two boundary value problems for X and Y. The nontrivial solutions are

$$X = X_m(x) = \sin(\mu_m x), \qquad \mu_m = \frac{m\pi}{a}, \qquad m \in \mathbb{N},$$
$$Y = Y_n(y) = \sin(\nu_n y), \qquad \nu_n = \frac{n\pi}{b}, \qquad n \in \mathbb{N},$$

with separation constants $B = -\mu_m^2$ and $C = -\nu_n^2$.

Since $T'' - c^2 AT = 0$, and $A = B + C = -\left(\mu_m^2 + \nu_n^2\right) < 0$,

$$T = T_{mn}(t) = B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t),$$

where

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

These are the **characteristic frequencies** of the membrane.

Normal modes

Assembling our results, we find that for any *pair* $m, n \in \mathbb{N}$ we have the *normal mode*

$$\begin{aligned} u_{mn}(x,y,t) &= X_m(x)Y_n(y)T_{mn}(t) \\ &= \sin(\mu_m x)\sin(\nu_n y)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right). \end{aligned}$$

Remarks:

- Note that the normal modes:
  - oscillate spatially with frequency μ_m in the x-direction,
  - oscillate spatially with frequency ν_n in the y-direction,
  - oscillate temporally with frequency λ_{mn}.
- While μ_m and ν_n are simply multiples of π/a and π/b, respectively, λ_{mn} *is not a multiple of any basic frequency.*

Superposition and initial conditions

Superposition gives the general solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(\mu_m x)\sin(\nu_n y)\left(B_{mn}\cos(\lambda_{mn}t) + B_{mn}^{*}\sin(\lambda_{mn}t)\right).$$

The initial conditions will determine the coefficients B_{mn} and B_{mn}^{*}. Setting $t = 0$ yields

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

$$g(x,y) = u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{mn}B_{mn}^{*}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right).$$

These are examples of *double Fourier series*.

Representability

Which functions are given by double Fourier series?

The following result partially answers this first question.

Theorem

If $f(x,y)$ is a C^2 function on the rectangle $[0,a]\times[0,b]$, then

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

for appropriate B_{mn}.

- To say that $f(x,y)$ is a C^2 *function* means that f as well as its first and second order partial derivatives are all continuous.
- While not as general as the Fourier representation theorem, this result is sufficient for our applications.

Orthogonality (again!)

How can we compute the coefficients in a double Fourier series?

The following result helps us answer this second question.

Theorem

The functions

$$Z_{mn}(x, y) = sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \quad m, n \in \mathbb{N}$$

are pairwise orthogonal relative to the inner product

$$\langle f, g\rangle = \int_0^a \int_0^b f(x, y)g(x, y)\, dy\, dx.$$

This is easily verified using the orthogonality of the functions $\sin(n\pi x/p)$ on the interval $[0, p]$.

Using the usual argument, it follows that if

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn} \underbrace{\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}_{Z_{mn}},$$

then

$$B_{mn} = \frac{\langle f, Z_{mn}\rangle}{\langle Z_{mn}, Z_{mn}\rangle} = \frac{\displaystyle\int_0^a\int_0^b f(x,y)Z_{mn}(x,y)\,dy\,dx}{\displaystyle\int_0^a\int_0^b Z_{mn}(x,y)^2\,dy\,dx}$$
$$= \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\,dy\,dx.$$

So, we can finally write down the complete solution to our original problem.

Conclusion

Theorem

Suppose that $f(x,y)$ and $g(x,y)$ are C^2 functions on the rectangle $[0,a]\times[0,b]$. The solution to the vibrating membrane problem is given by $u(x,y,t) =$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(\mu_m x)\sin(\nu_n y)\left(B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right)$$

where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2+\nu_n^2}$, and

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\mu_m x)\sin(\nu_n y)\,dy\,dx,$$

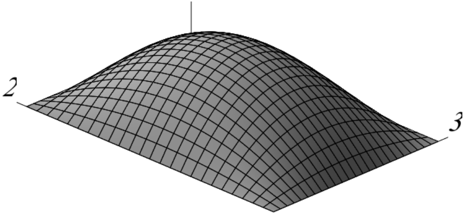
$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x,y)\sin(\mu_m x)\sin(\nu_n y)\,dy\,dx.$$

Example

A 2×3 *rectangular membrane has* $c = 6$. *If we deform it to have shape given by*

$$f(x, y) = xy(2 - x)(3 - y),$$

keep its edges fixed, and release it at $t = 0$, *find an expression that gives the shape of the membrane for* $t > 0$.



We must compute the coefficients B_{mn} and B^*_{mn}. Since $g(x,y) = 0$ we immediately have

$$B^*_{mn} = 0.$$

We also have

$$\begin{aligned} B_{mn} &= \frac{4}{2 \cdot 3} \int_0^2 \int_0^3 xy(2-x)(3-y) \sin\left(\frac{m\pi}{2}x\right) \sin\left(\frac{n\pi}{3}y\right)\, dy\, dx \\ &= \frac{2}{3} \int_0^2 x(2-x) \sin\left(\frac{m\pi}{2}x\right)\, dx \int_0^3 y(3-y) \sin\left(\frac{n\pi}{3}y\right)\, dy \\ &= \frac{2}{3} \left(\frac{16(1+(-1)^{m+1})}{\pi^3 m^3}\right) \left(\frac{54(1+(-1)^{n+1})}{\pi^3 n^3}\right) \\ &= \frac{576}{\pi^6} \frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3 n^3}. \end{aligned}$$

The coefficients λ_{mn} are given by

$$\lambda_{mn} = 6\pi\sqrt{\frac{m^2}{4} + \frac{n^2}{9}} = \pi\sqrt{9m^2 + 4n^2}.$$

Assembling all of these pieces yields

$$u(x,y,t) = \frac{576}{\pi^6}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3n^3}\sin\left(\frac{m\pi}{2}x\right)\right.$$
$$\left.\times\sin\left(\frac{n\pi}{3}y\right)\cos\left(\pi\sqrt{9m^2+4n^2}\,t\right)\right).$$

Example

Suppose in the previous example we also impose an initial velocity given by $g(x, y) = \sin 2\pi x$. Find an expression that gives the shape of the membrane for $t > 0$.

Since we have the same initial shape, B_{mn} don't change. We only need to find B^*_{mn} and add the appropriate terms to the previous solution.

Using λ_{mn} computed above, we have

$$\begin{aligned} B^*_{mn} &= \frac{2}{3\pi\sqrt{9m^2+4n^2}} \int_0^2 \int_0^3 \sin(2\pi x) \sin\left(\frac{m\pi}{2}x\right) \sin\left(\frac{n\pi}{3}y\right)\, dy\, dx \\ &= \frac{2}{3\pi\sqrt{9m^2+4n^2}} \int_0^2 \sin(2\pi x) \sin\left(\frac{m\pi}{2}x\right)\, dx \int_0^3 \sin\left(\frac{n\pi}{3}y\right)\, dy. \end{aligned}$$

The first integral is zero unless $m = 4$, i.e. $B^*_{mn} = 0$ for $m \neq 4$.

Evaluating the second integral, we have

$$B_{4n}^* = \frac{1}{3\pi\sqrt{36+n^2}}\frac{3(1+(-1)^{n+1})}{n\pi} = \frac{1+(-1)^{n+1}}{\pi^2 n\sqrt{36+n^2}}.$$

So the velocity dependent term of the solution is

$$\begin{aligned} u_2(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^* \sin(\mu_m x)\sin(\nu_n y)\sin(\lambda_{mn} t) \\ &= \frac{\sin(2\pi x)}{\pi^2}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n\sqrt{36+n^2}}\sin\left(\frac{n\pi}{3}y\right)\sin\left(2\pi\sqrt{36+n^2}\,t\right). \end{aligned}$$

If we let $u_1(x,y,t)$ denote the solution to the first example, the complete solution here is

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t).$$