Complete Series Solution of the Vibrating Circular Membrane Problem

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Partial Differential Equations
April 14, 2015
Normal modes of the vibrating circular membrane

Recall that for \( m \in \mathbb{N}_0, n \in \mathbb{N} \) these have the form

\[
J_m(\lambda_{mn}r) \left( A \cos (m\theta) + B \sin (m\theta) \right) \left( C \cos (c\lambda_{mn}t) + D \sin (c\lambda_{mn}t) \right),
\]

where \( \lambda_{mn} = \alpha_{mn}/a \), \( a > 0 \) is the radius of the membrane, and

\[
\alpha_{m1} < \alpha_{m2} < \alpha_{m3} < \cdots
\]

are the positive zeros of \( J_m(x) \). For convenience we set

\[
u_{mn}(r, \theta, t) = J_m(\lambda_{mn}r) (a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta)) \cos(c\lambda_{mn}t),
\]

\[
u^*_{mn}(r, \theta, t) = J_m(\lambda_{mn}r) (a^*_{mn} \cos(m\theta) + b^*_{mn} \sin(m\theta)) \sin(c\lambda_{mn}t),
\]

and use superposition to construct the general solution

\[
u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \nu_{mn}(r, \theta, t) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \nu^*_{mn}(r, \theta, t).
\]
In order to completely determine the shape of the membrane at any time we must specify the *initial conditions*

\[ u(r, \theta, 0) = f(r, \theta), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \text{ (shape)}, \]
\[ u_t(r, \theta, 0) = g(r, \theta), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \text{ (velocity)}. \]

Setting \( t = 0 \) in the general solution, we find that this requires

\[
f(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn}r) (a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta))
\]
\[
g(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c\lambda_{mn} J_m(\lambda_{mn}r) (a_{mn}^* \cos(m\theta) + b_{mn}^* \sin(m\theta))
\]

which are called *Fourier-Bessel expansions*. 

**Daileda**

**Solution of the Wave Equation on a Disk**
We will see later that the functions $R_{mn}(r) = J_m(\lambda_{mn}r)$ are orthogonal relative to the \textit{weighted inner product}

\[
\langle f, g \rangle = \int_0^a f(r)g(r)r \, dr.
\]

That is,

\[
\langle R_{mn}, R_{mk} \rangle = \int_0^a J_m(\lambda_{mn}r) J_m(\lambda_{mk}r) r \, dr = 0 \quad \text{if } n \neq k.
\]

In addition, it can also be shown that

\[
\langle R_{mn}, R_{mn} \rangle = \int_0^a J_m^2(\lambda_{mn}r) r \, dr = \frac{a^2}{2} J_{m+1}^2(\alpha_{mn}).
\]
Using the orthogonality relations for Bessel and trigonometric functions, one obtains:

**Theorem**

The functions

\[ \phi_{mn}(r, \theta) = J_m(\lambda_{mn} r) \cos(m\theta), \]
\[ \psi_{mn}(r, \theta) = J_m(\lambda_{mn} r) \sin(m\theta), \]

\((m \in \mathbb{N}_0, n \in \mathbb{N})\) form a (complete) orthogonal set of functions relative to the inner product

\[ \langle f, g \rangle = \int_0^{2\pi} \int_0^a f(r, \theta)g(r, \theta)r \, dr \, d\theta. \]

That is, \( \langle \phi_{mn}, \phi_{jk} \rangle = \langle \psi_{mn}, \psi_{jk} \rangle = 0 \) for \( (m, n) \neq (j, k) \) and \( \langle \phi_{mn}, \psi_{jk} \rangle = 0 \) for all \( (m, n) \) and \( (j, k) \).
Since our initial membrane shape condition is

\[ f(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (a_{mn}\phi_{mn}(r, \theta) + b_{mn}\psi_{mn}(r, \theta)), \]

the usual orthogonality argument gives

\[ a_{mn} = \frac{\langle f, \phi_{mn} \rangle}{\langle \phi_{mn}, \phi_{mn} \rangle} = \frac{\int_0^{2\pi} \int_0^a f(r, \theta)J_m(\lambda_{mn}r) \cos(m\theta) r \, dr \, d\theta}{\int_0^{2\pi} \int_0^a J_m^2(\lambda_{mn}r) \cos^2(m\theta) r \, dr \, d\theta}, \]

\[ b_{mn} = \frac{\langle f, \psi_{mn} \rangle}{\langle \psi_{mn}, \psi_{mn} \rangle} = \frac{\int_0^{2\pi} \int_0^a f(r, \theta)J_m(\lambda_{mn}r) \sin(m\theta) r \, dr \, d\theta}{\int_0^{2\pi} \int_0^a J_m^2(\lambda_{mn}r) \sin^2(m\theta) r \, dr \, d\theta}, \]

for \( m \geq 0, \ n \geq 1. \)
The integrals in the denominators can be evaluated explicitly:

\[
\int_{0}^{2\pi} \int_{0}^{a} J_{m}^{2}(\lambda_{mn}r) \cos^{2}(m\theta) \ r \ dr \ d\theta
\]

\[
= \int_{0}^{2\pi} \cos^{2}(m\theta) \ d\theta \int_{0}^{a} J_{m}^{2}(\lambda_{mn}r) \ r \ dr
\]

\[
= \begin{cases} 
\frac{\pi a^{2}}{2} J_{1}^{2}(\alpha_{0n}) & \text{if } m = 0, \\
\frac{\pi a^{2}}{2} J_{m+1}^{2}(\alpha_{mn}) & \text{if } m \geq 1;
\end{cases}
\]

and likewise

\[
\int_{0}^{2\pi} \int_{0}^{a} J_{m}^{2}(\lambda_{mn}r) \sin^{2}(m\theta) \ r \ dr \ d\theta = \frac{\pi a^{2}}{2} J_{m+1}^{2}(\alpha_{mn}),
\]

for \( m \geq 1 \).
We conclude that

\[ a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a f(r, \theta) J_0(\lambda_{0n}r) r \, dr \, d\theta, \]

\[ a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r, \theta) J_m(\lambda_{mn}r) \cos(m\theta) r \, dr \, d\theta, \]

\[ b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r, \theta) J_m(\lambda_{mn}r) \sin(m\theta) r \, dr \, d\theta, \]

for \( m, n \in \mathbb{N} \). Finally, recall the initial velocity condition

\[ g(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (c\lambda_{mn}a^*_\phi_{mn}(r, \theta) + c\lambda_{mn}b^*_\psi_{mn}(r, \theta)). \]
Integral formulae for $a_{mn}^*$ and $b_{mn}^*$

The same line of reasoning as above yields

$$a_{0n}^* = \frac{1}{\pi \alpha_{0n}\alpha_1} \int_0^{2\pi} \int_0^a g(r, \theta) J_0(\lambda_{0n}r) r \, dr \, d\theta,$$

$$a_{mn}^* = \frac{2}{\pi \alpha_{mn}\alpha_{m+1}} \int_0^{2\pi} \int_0^a g(r, \theta) J_m(\lambda_{mn}r) \cos(m\theta) r \, dr \, d\theta,$$

$$b_{mn}^* = \frac{2}{\pi \alpha_{mn}\alpha_{m+1}} \int_0^{2\pi} \int_0^a g(r, \theta) J_m(\lambda_{mn}r) \sin(m\theta) r \, dr \, d\theta,$$

for $m, n \in \mathbb{N}$.

This (essentially) completes the statement of the general solution to the vibrating circular membrane problem.
Since $\cos 0 = 1$ and $\sin 0 = 0$ we have

$$
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r) \left( a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta) \right) \cos(c\lambda_{mn} t)
$$

$$
= \sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r) \cos(c\lambda_{0n} t) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\text{as above})
$$

- Note that there are really no $b_{0n}$ coefficients.

- This is the "true form" of the first series in the solution.

Analogous comments hold for the second series.
If \( f(r, \theta) = f(r) \) (i.e. \( f \) is radially symmetric), then for \( m \neq 0 \)

\[
a_{mn} = (\cdots) \int_{0}^{2\pi} \int_{0}^{a} f(r) J_m(\lambda_{mn}r) \cos(m\theta) r \, dr \, d\theta \\
= (\cdots) \int_{0}^{a} \cdots \, dr \int_{0}^{2\pi} \cos(m\theta) \, d\theta = 0,
\]

and \( b_{mn} = 0 \), too. That is, there are only \( a_{0n} \) terms.

Likewise, if \( g \) is radially symmetric, then for \( m \neq 0 \)

\[
a_{mn}^* = b_{mn}^* = 0,
\]

and there are only \( a_{0n}^* \) terms.
Example

Solve the vibrating membrane problem with \( a = c = 1 \) and initial conditions

\[
f(r, \theta) = 1 - r^4, \quad g(r, \theta) = 0.
\]

Because \( g(r, \theta) = 0 \), we immediately find that \( a_{mn}^* = b_{mn}^* = 0 \) for all \( m \) and \( n \).

Because \( f \) is radially symmetric, we only need to compute \( a_{0n} \).

Since \( a = 1 \), \( \lambda_{mn} = \alpha_{mn} \), so

\[
a_{0n} = \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^1 f(r) J_0(\alpha_{0n}r) r \, dr \, d\theta
\]

\[
= \frac{2}{J_1^2(\alpha_{0n})} \int_0^1 (1 - r^4) J_0(\alpha_{0n}r) r \, dr
\]

substitute \( x = \alpha_{0n}r \)
\[
\frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \int_0^{\alpha_{0n}} \left( 1 - \frac{x^4}{\alpha_{0n}^4} \right) J_0(x) x \, dx
\]

\[
= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \left( \int_0^{\alpha_{0n}} xJ_0(x) \, dx - \frac{1}{\alpha_{0n}^4} \int_0^{\alpha_{0n}} x^5 J_0(x) \, dx \right).
\]

According to earlier results

\[
A = \int_0^{\alpha_{0n}} xJ_0(x) \, dx = xJ_1(x) \bigg|_0^{\alpha_{0n}} = \alpha_{0n}J_1(\alpha_{0n}),
\]

\[
B = \int_0^{\alpha_{0n}} x^5 J_0(x) \, dx = x^5 J_1(x) - 4x^4 J_2(x) + 8x^3 J_3(x) \bigg|_0^{\alpha_{0n}}
\]

\[
= \alpha_{0n}^5 J_1(\alpha_{0n}) - 4\alpha_{0n}^4 J_2(\alpha_{0n}) + 8\alpha_{0n}^3 J_3(\alpha_{0n}).
\]
It follows that

\[ a_{0n} = \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \left( A - \frac{1}{\alpha_{0n}^4} B \right) = \frac{8 \left( \alpha_{0n} J_2(\alpha_{0n}) - 2 J_3(\alpha_{0n}) \right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})}, \]

so that finally

\[ u(r, \theta, t) = \sum_{n=1}^{\infty} \frac{8 \left( \alpha_{0n} J_2(\alpha_{0n}) - 2 J_3(\alpha_{0n}) \right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})} J_0(\alpha_{0n} r) \cos(\alpha_{0n} t). \]

**Remark:** This solution can easily be implemented in Maple, since the command

\[ \text{BesselJZeros}(m,n) \]

will compute \( \alpha_{mn} \) numerically.
A non-symmetric example

Example

Solve the vibrating membrane problem with \( a = c = 1 \) and initial conditions:

\[
f(r, \theta) = r(1 - r^4) \cos \theta, \quad g(r, \theta) = 0.
\]

Since \( g \equiv 0 \), \( a^*_m = b^*_m = 0 \) for all \( m, n \). We also have

\[
b_{mn} = \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi} \int_0^1 r(1 - r^4) \cos \theta J_m(\alpha_{mn} r) \sin(m\theta) r \, dr \, d\theta
\]

\[
= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \left[ \int_0^{2\pi} \cos \theta \sin(m\theta) \, d\theta \right] \int_0^1 r(1 - r^4) J_m(\alpha_{mn} r) r \, dr
\]

\[
= 0 \quad \text{for all } m, n.
\]
Additionally,

\[
\begin{align*}
a_{0n} &= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^1 r(1 - r^4) \cos \theta J_0(\alpha_{0n} r) r \, dr \, d\theta \\
&= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi} \cos \theta \, d\theta \int_0^1 r(1 - r^4) J_0(\alpha_{0n} r) r \, dr \\
&= 0, \\
\end{align*}
\]

and

\[
\begin{align*}
a_{mn} &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^1 r(1 - r^4) \cos \theta J_m(\alpha_{mn} r) \cos(m\theta) r \, dr \, d\theta \\
&= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \cos \theta \cos(m\theta) \, d\theta \int_0^1 r(1 - r^4) J_m(\alpha_{mn} r) r \, dr.
\end{align*}
\]
The integral $A$ is zero unless $m = 1$, in which case it’s equal to $\pi$. In this case

$$a_{1n} = \frac{2}{J_2^2(\alpha_{1n})} \int_0^1 r(1 - r^4)J_1(\alpha_{1n}r)r \, dr$$

$$= \frac{2}{J_2^2(\alpha_{1n})} \left( \int_0^1 r^2 J_1(\alpha_{1n}r) \, dr - \int_0^1 r^6 J_1(\alpha_{1n}r) \, dr \right).$$

Substituting $x = \alpha_{1n}r$ and proceeding as before one can show

$$\int_0^1 r^2 J_1(\alpha_{1n}r) \, dr = \frac{J_2(\alpha_{1n})}{\alpha_{1n}},$$

$$\int_0^1 r^6 J_1(\alpha_{1n}r) \, dr = \frac{J_2(\alpha_{1n})}{\alpha_{1n}^2} - \frac{4J_3(\alpha_{1n})}{\alpha_{1n}^2} + \frac{8J_4(\alpha_{1n})}{\alpha_{1n}^3}.$$
Assembling these formulae gives

\[ a_{1n} = \frac{2}{J_2^2(\alpha_{1n})} \left( \frac{4J_3(\alpha_{1n})}{\alpha_{1n}^2} - \frac{8J_4(\alpha_{1n})}{\alpha_{1n}^3} \right) = \frac{8 \left( \alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n}) \right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})}. \]

Since all the other coefficients are zero,

\[ u(r, \theta, t) = \cos \theta \sum_{n=1}^{\infty} \frac{8 \left( \alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n}) \right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})} J_1(\alpha_{1n}r) \cos(\alpha_{1n}t). \]

**Remark:** In general, one should **not** expect the solution to reduce to a single series.
A "complicated" example

Example

Solve the vibrating membrane problem with \( a = 2, c = 1 \) and initial conditions

\[
    f(r, \theta) = 0, \quad g(r, \theta) = r^2(2 - r) \sin^8 \left( \frac{\theta}{2} \right).
\]

Since \( f \equiv 0 \), \( a_{mn} = 0 \), \( b_{mn} = 0 \). We also have

\[
    b^*_{mn} = (\cdots) \int_0^2 (\cdots) \, dr \int_0^{2\pi} \sin^8 \left( \frac{\theta}{2} \right) \sin(m\theta) \, d\theta = 0,
\]

odd, \( 2\pi \)-periodic

\[
    a^*_{0n} = \frac{1}{\pi \alpha_{0n}^2 J_1^2(\alpha_{0n})} \int_0^{2\pi} \sin^8 \left( \frac{\theta}{2} \right) \, d\theta \int_0^2 r^2(2 - r)J_0(\lambda_{0n}r) \, dr,
\]

\( 35\pi/64 \) (Maple)
and

\[ a_{mn}^* = \frac{2}{\pi \alpha_{mn} 2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \sin^8 \left( \frac{\theta}{2} \right) \cos(m\theta) \, d\theta \]

\[ \cdot \int_0^2 r^2 (2 - r) J_m(\lambda_{mn} r) r \, dr . \]

0 if \( m \geq 5 \) (Maple)

The solution therefore can be written

\[ u(r, \theta, t) = \sum_{m=0}^{4} \sum_{n=1}^{\infty} a_{mn}^* J_m(\lambda_{mn} r) \cos(m\theta) \sin(\lambda_{mn} t), \]

although the (?) integrals are not amenable to evaluation by hand.