# Weighted Inner Products, Sturm-Liouville Equations and Sturm-Liouville Theory

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# Inner products with weight functions

If $w(x) \geq 0$ for $x \in [a, b]$ we define the *inner product on* $[a, b]$ *with respect to the weight* $w$ to be

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\,dx.$$

We say $f$ and $g$ are *orthogonal on* $[a, b]$ *with respect to the weight* $w$ if

$$\langle f, g \rangle = 0.$$

**Remarks:**

- The inner product and orthogonality depend on the choice of $a$, $b$ and $w$.
- When $w(x) \equiv 1$, these definitions become the "ordinary" ones.
- Weighted inner products have exactly the same algebraic properties as the "ordinary" inner product.

# Examples

- The functions $f_n(x) = \sin(nx)$ $(n = 1, 2, \ldots)$ are pairwise orthogonal on $[0, \pi]$ with respect to the weight function $w(x) \equiv 1$.
- For a fixed $p \geq 0$, the functions $f_n(x) = J_p(\alpha_{pn}x/a)$ are pairwise orthogonal on $[0, a]$ with respect to the weight function $w(x) = x$.
- The functions

$$f_0(x) = 1, \quad f_1(x) = 2x, \quad f_2(x) = 4x^2 - 1, \quad f_3(x) = 8x^3 - 4x,$$
$$f_4(x) = 16x^4 - 12x^2 + 1, \quad f_5(x) = 32x^5 - 32x^3 + 6x$$

are pairwise orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$.

# Sturm-Liouville equations

A (second order) *Sturm-Liouville equation* has the form

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

where $p, q$ and $r$ are specific functions, and $\lambda$ is a parameter.

**Remarks:**

- Because $\lambda$ is a parameter, it is frequently replaced by other variables or expressions.
- Many "familiar" ODEs that occur during separation of variables can be put in Sturm-Liouville form.

## Example

*Show that $y'' + \lambda y = 0$ is a Sturm-Liouville equation.*

Take $p(x) = r(x) = 1$ and $q(x) = 0$.

## Example

*Put the parametric Bessel equation*

$$x^2y'' + xy' + (\lambda^2x^2 - m^2)y = 0$$

*in Sturm-Liouville form.*

First we divide by $x$ to get

$$\underbrace{xy'' + y'}_{(xy')'} + \left(\lambda^2 x - \frac{m^2}{x}\right) y = 0.$$

This is in Sturm-Liouville form with $p(x) = x$, $q(x) = -\frac{m^2}{x}$, $r(x) = x$, and parameter $\lambda^2$.

## Example

*Put* Chebyshev's differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0$$

*in Sturm-Liouville form.*

First we divide by $\sqrt{1-x^2}$ to get

$$\underbrace{\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}\,y'}_{(\sqrt{1-x^2}\,y')'} + \frac{n^2}{\sqrt{1-x^2}}\,y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = \sqrt{1-x^2}, \quad q(x) = 0, \quad r(x) = \frac{1}{\sqrt{1-x^2}},$$

provided we write the parameter as $n^2$.

# Sturm-Liouville problems

**Definition:** A (second order) *Sturm-Liouville (S-L) problem* consists of

- A Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b,$$

  together with

- *Boundary conditions*, i.e. specified behavior of $y$ at $x = a$ and $x = b$.

**Definition:** A function $y \not\equiv 0$ that solves an S-L problem is called an *eigenfunction*, and the corresponding value of $\lambda$ is called its *eigenvalue*.

# Examples

- The boundary value problem

$$y'' + \lambda y = 0, \quad y(-p) = y(p), \quad y'(-p) = y'(p),$$

is an S-L problem on the interval $[-p, p]$. We have seen that

$$\text{Eigenvalues: } \lambda = \lambda_n = \left(\frac{n\pi}{p}\right)^2 \quad (n \in \mathbb{N}_0)$$

$$\text{Eigenfunctions: } y = y_n = a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)$$

- The boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = y(L) = 0,$$

is an S-L problem on the interval $[0, L]$. We have seen that

$$\text{Eigenvalues: } \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad (n \in \mathbb{N})$$

$$\text{Eigenfunctions: } y = y_n = c_n \sin\left(\frac{n\pi x}{L}\right)$$

- The boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = -\kappa y(L),$$

is an S-L problem on the interval $[0, L]$. We have seen that

$$\text{Eigenvalues: } \lambda = \lambda_n = \mu_n^2 \quad \left(\tan \mu_n = -\frac{\mu_n}{\kappa}\right)$$

$$\text{Eigenfunctions: } y = y_n = c_n \sin(\mu_n x)$$

- The boundary value problem

$$x^2 y'' + xy' + (\lambda^2 x^2 - p^2) y = 0, \quad y(0) \text{ finite}, \quad y(a) = 0$$

is an S-L problem on the interval $[0, a]$. We have seen that

$$\text{Eigenvalues: } \lambda^2 = \lambda_n^2 = \left(\frac{\alpha_{pn}}{a}\right)^2 \quad (n \in \mathbb{N})$$

$$\text{Eigenfunctions: } y = y_n = c_n J_p\left(\frac{\alpha_{pn} x}{a}\right)$$

where $\alpha_{pn}$ is the $n$th positive zero of $J_p$.

# Inner products of eigenfunctions

Suppose $(y_j, \lambda_j)$, $(y_k, \lambda_k)$ are eigenfunction/eigenvalue pairs of an S-L problem:

$$\begin{aligned} (py_j')' + (q + \lambda_j r)y_j &= 0, \\ (py_k')' + (q + \lambda_k r)y_k &= 0. \end{aligned}$$

Multiply the first by $y_k$ and the second by $y_j$, then subtract to get

$$(py_j')'y_k - (py_k')'y_j + (\lambda_j - \lambda_k)y_j y_k r = 0.$$

Moving the $\lambda$-terms to one side and "adding zero," we get

$$\begin{aligned} (\lambda_j - \lambda_k)y_j y_k r &= (py_k')'y_j - (py_j')'y_k \\ &= (py_k')'y_j + py_k'y_j' - py_j'y_k' - (py_j')'y_k \\ &= (py_k'y_j - py_j'y_k)' \\ &= \left(p(y_k'y_j - y_j'y_k)\right)'. \end{aligned}$$

If $\lambda_j \neq \lambda_k$, we can divide by $\lambda_j - \lambda_k$ to get

$$y_j y_k r = \frac{\left(p(y_k' y_j - y_j' y_k)\right)'}{\lambda_j - \lambda_k} = \frac{(p \cdot W(y_j, y_k))'}{\lambda_j - \lambda_k}$$

Now integrate both sides to obtain:

## Proposition

*If $(y_j, \lambda_j)$, $(y_k, \lambda_k)$ are eigenfunction/eigenvalue pairs for an S-L problem on the interval $[a, b]$ and $\lambda_j \neq \lambda_k$, then their inner product with respect to the weight function $r(x)$ is*

$$\langle y_j, y_k \rangle = \int_a^b y_j(x) y_k(x) r(x)\, dx = \left. \frac{p(x) W(y_j, y_k)(x)}{\lambda_j - \lambda_k} \right|_a^b .$$

# Remarks

We will be interested in conditions that guarantee

$$\langle y_j, y_k \rangle = \left. \frac{p(x)W(y_j, y_k)(x)}{\lambda_j - \lambda_k} \right|_a^b = 0,$$

i.e. that eigenfunctions with distinct eigenvalues are *orthogonal*.

Examples include:

- *Periodic* S-L problems, i.e. $p(a) = p(b)$ and

$$y(a) = y(b), \quad y'(a) = y'(b).$$

- S-L problems satisfying $p(a) = 0$ and

$$y(a) \text{ is finite}, \quad y(b) = 0.$$

## Example

*Use the preceding results to establish the orthogonality of the trigonometric system*

$$\left\{1, \cos\left(\frac{\pi x}{p}\right), \cos\left(\frac{2\pi x}{p}\right), \ldots, \sin\left(\frac{\pi x}{p}\right), \sin\left(2\frac{\pi x}{p}\right), \ldots\right\}$$

*on the interval* $[-p, p]$ *with respect to the weight function* $w(x) = 1$.

The functions $y = \cos\left(\frac{n\pi x}{p}\right)$ and $y = \sin\left(\frac{n\pi x}{p}\right)$ are eigenfunctions of the periodic S-L problem

$$y'' + \lambda y = 0, \quad y(-p) = y(p), \quad y'(p) = y'(-p)$$

with eigenvalue $\lambda = \left(\frac{n\pi}{p}\right)^2$.

Since $r(x) \equiv 1$ in this case, we *automatically* find that

$$\int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right)\cos\left(\frac{n\pi x}{p}\right)\,dx = \int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx = \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx = 0 \quad \text{for } m \neq n.$$

We must verify orthogonality of $\cos\left(\frac{n\pi x}{p}\right)$ and $\sin\left(\frac{n\pi x}{p}\right)$ manually, since they have *the same eigenvalue*:

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx = \frac{p}{2n\pi}\sin^2\left(\frac{n\pi x}{p}\right)\Bigg|_{-p}^{p} = 0.$$

That covers every case, so we're done.

## Example

*Use the preceding results to establish the orthogonality of the Bessel function system*

$$\left\{ J_p\left(\frac{\alpha_{p1}x}{a}\right), J_p\left(\frac{\alpha_{p2}x}{a}\right), J_p\left(\frac{\alpha_{p3}x}{a}\right), \ldots, \right\}$$

*on the interval* $[0, a]$ *with respect to the weight function* $w(x) = x$.

The function $y = J_p\left(\frac{\alpha_{pn}x}{a}\right)$ is an eigenfunction of the S-L problem

$$x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \quad y(0) \text{ finite}, \quad y(a) = 0$$

with eigenvalue $\lambda^2 = \left(\frac{\alpha_{pn}}{a}\right)^2$. The S-L form of this equation is

$$(xy')' + \left(-\frac{p^2}{x} + \lambda^2 x\right) y = 0,$$

which shows that this problem is of the second type mentioned above. Since $r(x) = x$, we're done.

# Regular Sturm-Liouville problems

A *regular* Sturm-Liouville problem on $[a, b]$ has the form

$$\begin{aligned} (p(x)y')' + (q(x) + \lambda r(x))y &= 0, \quad a < x < b, \\ c_1 y(a) + c_2 y'(a) &= 0, \qquad (1) \\ d_1 y(b) + d_2 y'(b) &= 0, \qquad (2) \end{aligned}$$

where:

- $(c_1, c_2) \neq (0, 0)$ and $(d_1, d_2) \neq (0, 0)$;
- $p$, $p'$, $q$ and $r$ are continuous on $[a, b]$;
- $p$ and $r$ are positive on $[a, b]$.

**Remark:** We will focus on the boundary conditions (1) and (2), since they yield orthogonality of eigenfunctions.

# Examples

- The boundary value problem

$$y'' + \lambda y = 0, \qquad 0 < x < L,$$
$$y(0) = y(L) = 0,$$

is a regular S-L problem ($p(x) = 1$, $r(x) = 1$, $q(x) = 0$).

- The boundary value problem

$$((x^2 + 1)y')' + (x + \lambda)y = 0, \qquad -1 < x < 1,$$
$$y(-1) = y'(1) = 0,$$

is a regular S-L problem ($p(x) = x^2 + 1$, $q(x) = x$, $r(x) = 1$).

# Non-example

Although important, the boundary value problem

$$x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \qquad 0 < x < a,$$
$$y(0) \text{ finite, } y(a) = 0,$$

is *not* a regular Sturm-Liouville problem.

In Sturm-Liouville form we had $p(x) = r(x) = x$, $q(x) = -p^2/x$.

- $p$ and $r$ are *not positive* when $x = 0$.
- $q$ is *not continuous* when $x = 0$.
- The boundary condition at $x = 0$ is irregular.

This is an example of a *singular Sturm-Liouville problem.*

# Orthogonality in regular S-L problems

Suppose $y_j$, $y_k$ are eigenfunctions of a regular S-L problem with distinct eigenvalues $\lambda_j$, $\lambda_k$. The boundary condition at $x = a$ gives

$$\left.\begin{aligned} c_1 y_j(a) + c_2 y_j'(a) = 0, \\ c_1 y_k(a) + c_2 y_k'(a) = 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{pmatrix} y_j(a) & y_j'(a) \\ y_k(a) & y_k'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(c_1, c_2) \neq (0,0)$ the determinant must be zero:

$$y_j(a)y_k'(a) - y_k(a)y_j'(a) = W(y_j, y_k)(a) = 0.$$

Likewise, the boundary condition at $x = b$ gives $W(y_j, y_k)(b) = 0$. This means that

$$\langle y_j, y_k \rangle = \left. \frac{p(x)W(y_j, y_k)(x)}{\lambda_j - \lambda_k} \right|_a^b = 0.$$

# Eigenfunctions and eigenvalues of regular S-L problems

There's a great deal more that can be said about the eigenfunctions and eigenvalues of regular S-L problems.

**Theorem**

*The eigenvalues of a regular S-L problem on $[a, b]$ form an increasing sequence of real numbers*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

*with* $\lim_{n\to\infty} \lambda_n = \infty$.

*Eigenfunctions corresponding to distinct eigenvalues are orthogonal on $[a, b]$ with respect to the weight $r(x)$.*

*Moreover, the eigenfunction $y_n$ corresponding to $\lambda_n$ is unique (up to a scalar multiple), and has exactly $n - 1$ zeros in the interval $a < x < b$.*

# "Fourier convergence" for S-L problems

Without explicitly saying so, we have frequently made use of the following property of eigenfunctions of regular S-L problems.

**Theorem**

*Let $y_1, y_2, y_3, \ldots$ be the eigenfunctions of a (regular) S-L problem on $[a, b]$. If $f$ is piecewise smooth on $[a, b]$, then*

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} A_n y_n(x),$$

*where*

$$A_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\displaystyle\int_a^b f(x) y_n(x) r(x)\, dx}{\displaystyle\int_a^b y_n^2(x) r(x)\, dx}.$$

# Remarks

- The series $\sum_{n=1}^{\infty} A_n y_n$ is the *eigenfunction expansion* of $f$. Recall that $f(x) = \frac{f(x+)+f(x-)}{2}$ anywhere $f$ is continuous, so the eigenfunction expansion is equal to $f$ at most points.
- We have already proven the orthogonality statement, and have many times used it to "extract" the coefficients in eigenfunction expansions. Proofs of the remaining results are beyond the scope of our class.
- Aside from the "moreover" statement, these theorems hold for many irregular S-L problems (as we have seen). The "original" Fourier convergence theorem is one such example.
- The results of S-L theory unify and explain all of the ODE boundary value problems we have encountered throughout the semester!