The Fourier Transform

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The Fourier series representation

For periodic functions

Recall: If f is a $2p$-periodic (piecewise smooth) function, then f can be expressed as a sum of sinusoids with frequencies $0, 1/p, 2/p, 3/p, \ldots$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right),$$

where

$$a_n = \underbrace{\frac{1}{p}}_{2p \text{ when } n=0} \int_{-p}^{p} f(t) \overbrace{\cos}^{\text{sine for } b_n} \left(\frac{n\pi t}{p}\right) dt.$$

We can obtain a similar representation for arbitrary (non-periodic) functions, provided we replace the *(discrete) sum* with a *(continuous) integral*.

The Fourier integral representation

For $L^1(\mathbb{R})$ functions

Theorem

If f is piecewise smooth and $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, then

$$f(x) = \int_0^\infty \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)\,d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$$

Remarks:

- When $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, we say that $f \in L^1(\mathbb{R})$ (*integrable*).
- The integral actually equals $\frac{f(x+)+f(x-)}{2}$, which is "almost" f.

Example

If $a > 0$, find the Fourier integral representation for

$$f(x) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Note that f is piecewise smooth and that

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-a}^{a} dx = 2a < \infty,$$

so f has an integral representation.
We find that

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt = \frac{1}{\pi}\int_{-a}^{a} \cos(\omega t)\, dt$$
$$= \frac{1}{\pi}\frac{\sin(\omega t)}{\omega}\bigg|_{t=-a}^{t=a} = \frac{2\sin(\omega a)}{\pi\omega}.$$

Likewise

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt = \frac{1}{\pi}\int_{-a}^{a} \underbrace{\sin(\omega t)}_{\text{odd}}\,dt = 0.$$

It follows that the integral representation of f is

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin(\omega a)\cos(\omega x)}{\omega}\,d\omega = \frac{f(x+)+f(x-)}{2} = \begin{cases} 1 & \text{if } |x| < a, \\ \frac{1}{2} & \text{if } |x| = a, \\ 0 & \text{otherwise.} \end{cases}$$

Remarks:

- It is difficult to evaluate the integral on the left directly; we have indirectly produced a formula for its value.
- Expressing a difficult integral as the Fourier representation of a “familiar” function gives a powerful technique for evaluating definite integrals.

"Complexifying" the Fourier integral representation

From Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ one can deduce

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

If we apply these in the Fourier integral representation we obtain

$$\begin{aligned}
f(x) &= \int_0^\infty \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)\,d\omega \\
&= \frac{1}{2}\int_0^\infty \left(A(\omega) - iB(\omega)\right)e^{i\omega x}\,d\omega + \underbrace{\frac{1}{2}\int_0^\infty \left(A(\omega) + iB(\omega)\right)e^{-i\omega x}\,d\omega}_{\text{sub.}-\omega\text{ for }\omega} \\
&= \frac{1}{2}\int_0^\infty \left(A(\omega) - iB(\omega)\right)e^{i\omega x}\,d\omega + \frac{1}{2}\int_{-\infty}^0 (\underbrace{A(-\omega)}_{\text{even}} + i\,\underbrace{B(-\omega)}_{\text{odd}})e^{i\omega x}\,d\omega \\
&= \frac{1}{2}\int_{-\infty}^\infty \left(A(\omega) - iB(\omega)\right)e^{i\omega x}\,d\omega
\end{aligned}$$

The Fourier transform

According to their definitions, we have

$$\begin{aligned} A(\omega) - iB(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\left(\cos(\omega x) - i\sin(\omega x)\right)\,dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx. \end{aligned}$$

Definition: The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \sqrt{\frac{\pi}{2}}(A(\omega) - iB(\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Note that the Fourier integral representation now becomes

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega,$$

which gives the *inverse Fourier transform*.

Example

For $a > 0$, find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

We already found that $A(\omega) = \frac{2\sin(\omega a)}{\pi\omega}$ and $B(\omega) = 0$, so

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{2}}(A(\omega) - iB(\omega)) = \sqrt{\frac{2}{\pi}}\frac{\sin(\omega a)}{\omega}.$$

Remark: As this example demonstrates, if one already knows the Fourier integral representation of f, finding $\hat{f}$ is easy.

Example

For $a > 0$, find the Fourier transform of

$$f(x) = \begin{cases} -1 & \text{if } -a < x < 0, \\ 1 & \text{if } 0 < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left(-\int_{-a}^{0} e^{-i\omega x}\, dx + \int_{0}^{a} e^{-i\omega x}\, dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{i\omega} e^{-i\omega x}\Big|_{-a}^{0} - \frac{1}{i\omega} e^{-i\omega x}\Big|_{0}^{a} \right) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i\omega} \left(2 - e^{i\omega a} - e^{-i\omega a}\right) \\ &= \frac{1}{i\omega\sqrt{2\pi}} (2 - 2\cos(\omega a)) = i\sqrt{\frac{2}{\pi}} \frac{(\cos(\omega a) - 1)}{\omega}. \end{aligned}$$

Example

Find the Fourier transform of $f(x) = e^{-|x|}$.

We have

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x} e^{-i\omega x}\, dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x(1-i\omega)}\, dx + \int_{0}^{\infty} e^{-x(1+i\omega)}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left. \frac{e^{x(1-i\omega)}}{1-i\omega} \right|_{-\infty}^{0} - \left. \frac{e^{-x(1+i\omega)}}{1+i\omega} \right|_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}.
\end{aligned}$$

Properties of the Fourier transform

Linearity: If $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$, then

$$\widehat{af + bg} = a\hat{f} + b\hat{g}.$$

Transforms of Derivatives: If $f, f', f'', \ldots, f^{(n)} \in L^1(\mathbb{R})$ and $f^{(k)}(x) \to 0$ as $|x| \to \infty$ for $k = 1, 2, \ldots, n-1$, then

$$\widehat{f^{(n)}}(\omega) = (i\omega)^n \hat{f}(\omega),$$

i.e. differentiation becomes multiplication by $i\omega$.

Derivatives of Transforms: If $f, x^n f \in L^1(\mathbb{R})$, then

$$\hat{f}^{(n)}(\omega) = \frac{1}{i^n}\widehat{x^n f}(\omega),$$

i.e. multiplication by x (almost) becomes differentiation.

Remarks

- Linearity follows immediately from the definition and linearity of integration.
- For the second fact, if $f, f' \in L^1(\mathbb{R})$ then

$$\widehat{f'}(\omega) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx}_{u=e^{-i\omega x},\, dv=f'(x)\,dx}$$

$$= \frac{1}{\sqrt{2\pi}} \left(f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \right) = i\omega \hat{f}(\omega),$$

provided $f(\pm\infty) = 0$. The general result follows by iteration, e.g.

$$\widehat{f''}(\omega) = \widehat{(f')'}(\omega) = i\omega\widehat{f'}(\omega) = (i\omega)^2\hat{f}(\omega).$$

- The final fact follows by "differentiating under the integral."

Example

Find the Fourier transform of $f(x) = (1 - x^2)e^{-|x|}$.

Using properties of the Fourier transform, we have

$$\widehat{f} = \widehat{e^{-|x|}} - \widehat{x^2 e^{-|x|}} = \widehat{e^{-|x|}} - i^2 \widehat{e^{-|x|}}''$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2} + \left(\frac{1}{1+\omega^2} \right)'' \right)$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2} + \frac{6\omega^2 - 2}{(1+\omega^2)^3} \right) = \sqrt{\frac{2}{\pi}} \frac{\omega^4 + 8\omega^2 - 1}{(1+\omega^2)^3}.$$

Remark: The Fourier inversion formula tells us that

$$\int_{-\infty}^{\infty} \frac{\omega^4 + 8\omega^2 - 1}{(1+\omega^2)^3} e^{i\omega x}\, d\omega = \pi(1 - x^2)e^{-|x|}.$$

Imagine trying to show this directly!

Example

Find the Fourier transform of the Gaussian function $f(x) = e^{-x^2}$.

Start by noticing that $y = f(x)$ solves $y' + 2xy = 0$. Taking Fourier transforms of both sides gives

$$(i\omega)\hat{y} + 2i\hat{y}' = 0 \quad \Rightarrow \quad \hat{y}' + \frac{\omega}{2}\hat{y} = 0.$$

The solutions of this (separable) differential equation are

$$\hat{y} = Ce^{-\omega^2/4}.$$

We find that

$$C = \hat{y}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-i0x}\, dx = \frac{1}{\sqrt{2\pi}}\sqrt{\pi} = \frac{1}{\sqrt{2}},$$

and hence $\hat{y} = \hat{f}(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$.

Example (Another useful property)

Show that for any $a \neq 0$, $\widehat{f(ax)} = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$.

We simply compute

$$\widehat{f(ax)} = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}\,dx}_{\text{sub. } u=ax} = \frac{1}{|a|\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{-i\omega u/a}\,du = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$$

Example

Find the Fourier transform of $f(x) = e^{-x^2/2}$.

Since $f(x) = e^{-(x/\sqrt{2})^2}$, taking $a = 1/\sqrt{2}$ above and using the Gaussian example gives

$$\hat{f}(\omega) = \sqrt{2}\frac{1}{\sqrt{2}}e^{-(\sqrt{2}\omega)^2/4} = e^{-\omega^2/2}.$$

Example

Find the general solution to the ODE $y'' - y = e^{-x^2/2}$.

The corresponding homogeneous equation is

$$y_h'' - y_h = 0 \quad \Rightarrow \quad y_h = c_1 e^x + c_2 e^{-x}.$$

It remains to find a particular solution y_p.
We assume that an $L^1(\mathbb{R})$ solution exists, and take the Fourier transform of the original ODE:

$$(i\omega)^2 \hat{y} - \hat{y} = e^{-\omega^2/2} \quad \Rightarrow \quad \hat{y} = \frac{-e^{-\omega^2/2}}{\omega^2 + 1}.$$

Applying the inverse Fourier transform we obtain

$$y_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-e^{-\omega^2/2}}{\omega^2 + 1} e^{i\omega x}\, d\omega.$$

We can eliminate i by a judicious use of symmetry:

$$y_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{-e^{-\omega^2/2}}{\omega^2+1}\cos(\omega x)}_{\text{even}}\, d\omega + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{-e^{-\omega^2/2}}{\omega^2+1}\sin(\omega x)}_{\text{odd}}\, d\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{-e^{-\omega^2/2}}{\omega^2+1}\cos(\omega x)\, d\omega.$$

Hence, the complete solution is

$$y = y_h + y_p = c_1 e^x + c_2 e^{-x} + \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{-e^{-\omega^2/2}}{\omega^2+1}\cos(\omega x)\, d\omega.$$

A common feature of the *Fourier series method* for solving DEs is that the solutions must frequently be expressed as integrals.

Convolution of functions

Given two functions f and g (with domain $\mathbb{R}$), their *convolution* is the function $f * g$ defined by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)\,dt.$$

Remarks.

- If $f, g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ as well.
- The substitution $t \mapsto x - t$ can be used to show that

$$f * g = g * f.$$

- In general, there is no "easy" interpretation of the convolution.

Example

Suppose that if $f \in L^1(\mathbb{R})$ *is even, and let* $g(x) = \sin(ax)$*. Show that*

$$(f * g)(x) = \widehat{f}(a)\sin(ax)$$

According to the definition of convolution

$$\begin{aligned}(g * f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)f(t)\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(a(x-t))f(t)\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ia(x-t)} - e^{-ia(x-t)}}{2i} f(t)\,dt \\ &= \frac{1}{2i\sqrt{2\pi}} \left(e^{iax} \int_{-\infty}^{\infty} f(t)e^{-iat}\,dt - e^{-iax} \int_{-\infty}^{\infty} f(t)e^{iat}\,dt \right).\end{aligned}$$

Since t is even, we can substitute $-t$ for t in the second integral to get

$$(g * f)(x) = \frac{1}{2i\sqrt{2\pi}} \left(e^{iax} \int_{-\infty}^{\infty} f(t) e^{-iat}\,dt - e^{-iax} \int_{-\infty}^{\infty} f(t) e^{-iat}\,dt \right)$$

$$= \frac{1}{2i} \left(e^{iax} \widehat{f}(a) - e^{-iax} \widehat{f}(a) \right)$$

$$= \widehat{f}(a)\, \frac{e^{iax} - e^{-iax}}{2i} = \widehat{f}(a) \sin(ax).$$

Since $g * f = f * g$, this is what we needed to show.

Although convolution itself may seem somewhat "unnatural," it interacts naturally with the Fourier transform.

Theorem

If $f, g \in L^1(\mathbb{R})$, then
$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

Proof. We have

$$\begin{aligned}\widehat{f * g}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x) e^{-i\omega x}\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-t) g(t) e^{-i\omega x}\, dt\, dx\end{aligned}$$

Setting $u = x - t$ in the inner integral, so that $du = -dt$ and $x = t + u$, gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(t) e^{-i\omega t} e^{-i\omega u}\, du\, dx = \widehat{f}(\omega) \widehat{g}(\omega).$$