An Introduction to Bessel Functions

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Partial Differential Equations
April 9, 2015

Bessel's equation

Given $p \geq 0$, the ordinary differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad x > 0$$

is known as *Bessel's equation of order p*. In standard form this has

$$\left.\begin{aligned} p(x) &= \frac{1}{x}, \\ q(x) &= \frac{x^2 - p^2}{x^2} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} xp(x) &= 1, \\ x^2q(x) &= x^2 - p^2. \end{aligned}$$

so that $x = 0$ is a regular singularity with indicial equation

$$r^2 + (1-1)r - p^2 = 0 \;\; \Rightarrow \;\; r = \pm p.$$

The method of Frobenius

Consequently, for $r = p$ at least, we know there is a solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n} \qquad (a_0 \neq 0),$$

which will converge for all $x > 0$.

Substituting this into Bessel's equation and collecting terms with common powers of x gives

$$a_0(r^2 - p^2)x^r + a_1\left((r+1)^2 - p^2\right)x^{r+1} + \sum_{m=2}^{\infty}\left(a_m\left((r+m)^2 - p^2\right) + a_{m-2}\right)x^{r+m} = 0.$$

Setting the coefficients equal to zero gives the equations

$$a_0(r^2 - p^2) = 0 \quad \underset{a_0 \neq 0}{\Rightarrow} \quad r = \pm p,$$
$$a_1\left((r+1)^2 - p^2\right) = 0 \quad \Rightarrow \quad a_1 = 0,$$
$$a_m = \frac{-a_{m-2}}{(r+m)^2 - p^2} = \frac{-a_{m-2}}{m(m+2r)} \quad (m \geq 2).$$

These imply that $a_1 = a_3 = a_5 = \cdots = a_{2k+1} = 0$ and, taking $r = p$,

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1+p)(2+p)\cdots(k+p)}.$$

This gives the first Frobenius solution

$$y_1 = x^p \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{2^{2k} k!(1+p)(2+p)\cdots(k+p)} x^{2k}$$

Interlude

The Gamma function

In this case, the standard choice for a_0 involves the *Gamma function*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \qquad (s > 0).$$

One can use integration by parts to show that

$$\Gamma(s+1) = s\,\Gamma(s).$$

Applying this repeatedly, we find that for $k \in \mathbb{N}$

$$\begin{aligned}
\Gamma(s+k) &= (s+k-1)\Gamma(s+k-1) \\
&= (s+k-1)(s+k-2)\Gamma(s+k-2) \\
&= (s+k-1)(s+k-2)(s+k-3)\Gamma(s+k-3) \\
&\ \,\vdots \\
&= (s+k-1)(s+k-2)(s+k-3)\cdots s\,\Gamma(s).
\end{aligned}$$

This has two nice consequences.

- One can show $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$, so setting $s = 1$ above:

$$\Gamma(k+1) = k(k-1)(k-2)\cdots 1 \cdot \Gamma(1) = k!$$

  This is why $\Gamma(s)$ is called the *generalized factorial*.

- Setting $s = p + 1$ above:

$$\Gamma(p+1+k) = (p+k)(p+k-1)\cdots(p+1)\Gamma(p+1)$$

  or

$$\frac{1}{(1+p)(2+p)\cdots(k+p)} = \frac{\Gamma(p+1)}{\Gamma(k+p+1)}.$$

Bessel functions of the first kind

Returning to Bessel's equation, we find that the first Frobenius solution can be written

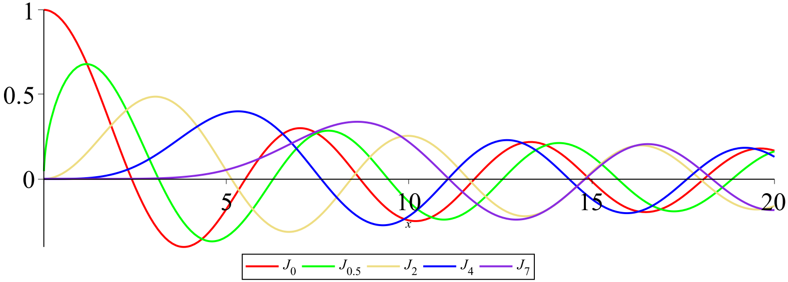
$$y_1 = x^p \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{2^{2k} k!(1+p)(2+p)\cdots(k+p)} x^{2k}$$

$$= 2^p \Gamma(p+1) a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}.$$

Taking $a_0 = \frac{1}{2^p \Gamma(p+1)}$ yields the *Bessel function of the first kind of order* p:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}.$$

Graphs of Bessel functions of the first kind



In Maple, the functions $J_p(x)$ can be invoked by the command

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BesselJ(p,x)
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Properties of Bessel functions of the first kind

- $J_0(0) = 1$ and $J_p(0) = 0$ for $p > 0$.
- The values of J_p always lie between 1 and -1.
- J_p has infinitely many positive zeros, which we denote by

$$0 < \alpha_{p1} < \alpha_{p2} < \alpha_{p3} < \cdots$$

- J_p is oscillatory and tends to zero as $x \to \infty$. More precisely,

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right).$$

- $\lim_{n\to\infty} |\alpha_{pn} - \alpha_{p,n+1}| = \pi$.

- For $0 < p < 1$, the graph of J_p has a vertical tangent line at $x = 0$.
- For $1 < p$, the graph of J_p has a horizontal tangent line at $x = 0$, and the graph is initially "flat."
- For some values of p, the Bessel functions of the first kind can be expressed in terms of familiar functions, e.g.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$
$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right).$$

Remarks

- Frobenius' method yields a second linearly independent solution y_2 of Bessel's equation.
- Although the exact form of y_2 depends on the value of p, it is not hard to argue that in any case $\lim_{x\to 0^+} |y_2| = \infty$.
- Since $\lim_{x\to 0^+} J_p(x)$ is finite, it follows that *any* linearly independent solution $Y_p(x)$ must also satisfy
$$\lim_{x\to 0^+} |Y_p(x)| = \infty.$$
- The standard normalization of Y_p is called the *Bessel function of the second kind*. We won't explicitly need it.

Differentiation identities

Using the series definition of $J_p(x)$, one can show that:

$$\begin{aligned} \frac{d}{dx}\left(x^p J_p(x)\right) &= x^p J_{p-1}(x), \\ \frac{d}{dx}\left(x^{-p} J_p(x)\right) &= -x^{-p} J_{p+1}(x). \end{aligned} \tag{1}$$

The product rule and cancellation lead to

$$\begin{aligned} xJ_p'(x) + pJ_p(x) &= xJ_{p-1}(x), \\ xJ_p'(x) - pJ_p(x) &= -xJ_{p+1}(x). \end{aligned}$$

Addition and subtraction of these identities then yield

$$\begin{aligned} J_{p-1}(x) - J_{p+1}(x) &= 2J_p'(x), \\ J_{p-1}(x) + J_{p+1}(x) &= \frac{2p}{x} J_p(x). \end{aligned}$$

Integration identities

Integration of the differentiation identities (1) gives

$$\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C$$
$$\int x^{-p+1} J_p(x)\,dx = -x^{-p+1} J_{p-1}(x) + C.$$

- Exercises 4.2.12 and 4.3.9 give similar identities.
- Identities such as these can be used to evaluate certain integrals of the form

$$\int_0^a f(r) J_m(\lambda_{mn} r) r\,dr,$$

which will occur frequently in later work.

Example

Evaluate

$$\int x^{p+5} J_p(x)\, dx.$$

We integrate by parts, first taking

$$\begin{aligned} u &= x^4 & dv &= x^{p+1} J_p(x)\, dx \\ du &= 4x^3\, dx & v &= x^{p+1} J_{p+1}(x), \end{aligned}$$

which gives

$$\int x^{p+5} J_p(x)\, dx = x^{p+5} J_{p+1}(x) - 4 \int x^{p+4} J_{p+1}(x)\, dx.$$

Now integrate by parts again with

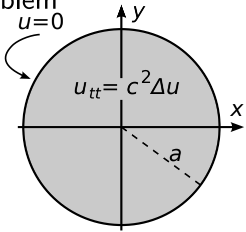
$$u = x^2 \qquad dv = x^{p+2} J_{p+1}(x)\,dx$$
$$du = 2x\,dx \qquad v = x^{p+2} J_{p+2}(x),$$

to get

$$\begin{aligned}\int x^{p+5} J_p(x)\,dx &= x^{p+5} J_{p+1}(x) - 4\int x^{p+4} J_{p+1}(x)\,dx \\ &= x^{p+5} J_{p+1}(x) - 4\left(x^{p+4} J_{p+2}(x) - 2\int x^{p+3} J_{p+2}(x)\,dx\right) \\ &= x^{p+5} J_{p+1}(x) - 4x^{p+4} J_{p+2}(x) + 8x^{p+3} J_{p+3}(x) + C.\end{aligned}$$

Return of the vibrating circular membrane

Recall that the vibrating circular membrane problem



$$u_{tt} = c^2 \Delta u = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right),$$
$$0 < r < a, \ \ 0 < \theta < 2\pi, \ \ t > 0,$$

$$u(a, \theta, t) = 0, \ \ 0 \le \theta \le 2\pi, \ \ t > 0,$$

led to the separated ODE boundary value problem

$$r^2 R'' + rR' + \left(\lambda^2 r^2 - \mu^2\right) R = 0, \quad R(0+) \text{ finite}, \quad R(a) = 0,$$
$$\Theta'' + \mu^2 \Theta = 0, \quad \Theta \ \ 2\pi\text{-periodic},$$
$$T'' + c^2 \lambda^2 T = 0,$$

and that the solutions to the Θ problem are

$$\Theta(\theta) = \Theta_m(\theta) = A\cos(m\theta) + B\sin(m\theta), \quad \mu = m \in \mathbb{N}_0.$$

In the homework you showed that $\lambda = 0$ implies $R \equiv 0$, so we are faced with solving the *parametric Bessel equation*

$$r^2 R'' + rR' + (\lambda^2 r^2 - m^2)R = 0 \quad (\lambda > 0) \tag{2}$$

subject to the boundary conditions

$$R(0+) \text{ finite}, \quad R(a) = 0.$$

If we let $x = \lambda r$, then the chain rule implies

$$R' = \frac{dR}{dr} = \frac{dR}{dx}\frac{dx}{dr} = \lambda \dot{R},$$
$$R'' = \frac{dR'}{dr} = \lambda \frac{d\dot{R}}{dr} = \lambda \frac{d\dot{R}}{dx}\frac{dx}{dr} = \lambda^2 \ddot{R}.$$

Hence (2) becomes

$$x^2 \ddot{R} + x\dot{R} + (x^2 - m^2)R = 0,$$

which is Bessel's equation of order m.

It follows that

$$R = c_1 J_m(x) + c_2 Y_m(x) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r).$$

Because $\lim_{\xi \to 0^+} |Y_m(\xi)| = \infty$, we find that

$$\begin{aligned}
R(0+) \text{ finite} \quad &\Rightarrow \quad c_2 = 0 \quad \Rightarrow \quad R = c_1 J_m, \\
R(a) = 0 \quad &\Rightarrow \quad R(a) = c_1 J_m(\lambda a) = 0 \quad \underset{c_1 \neq 0}{\Rightarrow} \quad J_m(\lambda a) = 0 \\
&\Rightarrow \quad \lambda a = \alpha_{mn}, \qquad n \in \mathbb{N} \\
&\Rightarrow \quad \lambda = \lambda_{mn} = \frac{\alpha_{mn}}{a}, \qquad n \in \mathbb{N}
\end{aligned}$$

Choosing $c_1 = 1$, we find that

$$R(r) = R_{mn}(r) = J_m(\lambda_{mn} r) = J_m\left(\frac{\alpha_{mn} r}{a}\right) \qquad m \in \mathbb{N}_0, \quad n \in \mathbb{N}.$$

Normal modes of the vibrating circular membrane

Returning to T (which solves $T'' + c^2\lambda^2 T = 0$), we finally find

$$T(t) = T_{mn}(t) = C\cos\left(c\lambda_{mn}t\right) + D\sin\left(c\lambda_{mn}t\right).$$

and arrive at the normal modes for the vibrating circular membrane: $u_{mn}(r,\theta,t) = R_{mn}(r)\Theta_m(\theta)T_{mn}(t) =$

$$J_m\left(\lambda_{mn}r\right)\left(A\cos\left(m\theta\right) + B\sin\left(m\theta\right)\right)\left(C\cos\left(c\lambda_{mn}t\right) + D\sin\left(c\lambda_{mn}t\right)\right),$$

for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$.

Note that, up to scaling, rotation and a phase shift in time, these have the form

$$u(r,\theta,t) = J_m(\lambda_{mn}r)\,\cos(m\theta)\,\cos(c\lambda_{mn}t).$$