EXERCISE 4.3.3

Proposition. *Let* $a_1, a_2, a_3, \ldots, a_{2n+1}$ *be elements of a commutative ring. If*

$$S_k = \sum_{j=1}^{k} (-)^{j+1} a_j.^*$$

then

$$\sum_{j=1}^{2n+1} (-)^{j+1} a_j^2 = S_{2n+1}^2 + 2\sum_{j=1}^{n} S_{2j}(a_{2j} - a_{2j+1}). \tag{1}$$

Proof. We apply summation by part to the left hand side of the identity. This yields

$$\begin{aligned}
\sum_{j=1}^{2n+1} (-)^{j+1} a_j^2 &= \sum_{j=1}^{2n+1} \left((-)^{j+1} a_j\right) a_j \\
&= \sum_{j=1}^{2n+1} (S_j - S_{j-1})\, a_j \\
&= \sum_{j=1}^{2n+1} S_j a_j - \sum_{j=1}^{2n+1} S_{j-1} a_j \\
&= S_{2n+1} a_{2n+1} + \sum_{j=1}^{2n} S_j a_j - \sum_{j=1}^{2n} S_j a_{j+1} \\
&= S_{2n+1}(S_{2n+1} - S_{2n}) + \sum_{j=1}^{2n} S_j (a_j - a_{j+1}) \\
&= S_{2n+1}^2 - S_{2n+1} S_{2n} + \sum_{j=1}^{2n} S_j (a_j - a_{j+1}).
\end{aligned} \tag{2}$$

Now notice that for any $k \geq 1$

$$\begin{aligned}
S_{k+1} S_k &= \left(S_k + (-)^k a_{k+1}\right)\left(S_{k-1} + (-)^{k+1} a_k\right) \\
&= S_k S_{k-1} + (-)^{k+1} S_k a_k + (-)^k S_{k-1} a_{k+1} - a_k a_{k+1}{}^{**} \\
&= S_k S_{k-1} + (-)^{k+1} S_k a_k + S_k (-)^k a_{k+1} \\
&= S_k S_{k-1} + (-)^{k+1} S_k (a_k - a_{k+1})
\end{aligned}$$

Applying this successively to $S_{2n+1}S_{2n}$, $S_{2n}S_{2n-1}$, $S_{2n-1}S_{2n-2}, \ldots$, $S_2 S_1$ in (2) will alternately double an even indexed term in the sum and then cancel an odd indexed one, leaving us with the right hand side of (1) plus $S_1 S_0$. Since $S_0 = 0$, this completes the proof. ☐

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*We have used the somewhat awkward notation $(-)^k b$ to simply indicate k negations of b. In particular, $(-)^k b = b$ if k is even, while $(-)^k b = -b$ when k is odd. If we knew that the ring under consideration had an identity, a more natural way to write this would be the more familiar $(-1)^k b$.

**This is the only point at which we are forced to use commutativity.

Exercise 3. Prove that for any decreasing sequence $a_1, a_2, \ldots, a_{2n+1}$ of real numbers one has

$$a_1^2 - a_2^2 + a_3^2 - \cdots + a_{2n+1}^2 \geq (a_1 - a_2 + a_3 - \cdots + a_{2n+1})^2.$$

Proof. This is immediate from the result above since all of the terms in the sum on the right hand side of (1) are nonnegative in this case. $\square$