CHAPTER 10

METHOD OF CHARACTERISTICS

10.1 INTRODUCTION TO THE METHOD OF CHARACTERISTICS

The **method of characteristics**, which we develop now, is quite different from the methods we have studied so far. The method of characteristics is mathematically rather simple; in fact, this material could have been covered much earlier. It involves an interesting logical twist, and to explain it, we introduce the idea of an **anchor point**, a concept we find very useful but is not used in other textbooks.

We begin with a simple first-order partial differential equation,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{10.1}$$

$$u(x, 0) = f(x).$$

Notice that no boundary condition is needed, since we are working on the entire real line. As seen earlier, a solution of a partial differential equation for a function of two variables, $u(x, t)$, can be understood as a surface over the xt-plane. To parametrize this surface, we use **characteristic curves**, shown in Figure 10.1. We want to find curves $x(t)$ in the xt-plane such that the PDE can be reduced to an ODE along these

Partial Differential Equations: Theory and Completely Solved Problems, First Edition. By T. Hillen, I. E. Leonard, H. van Roessel

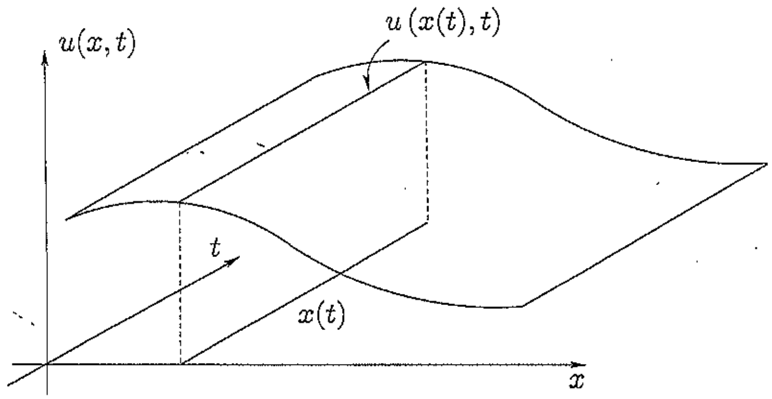


Figure 10.1 Schematic of a typical characteristic curve $x(t)$ and the part $u(x(t), t)$ of the solution surface u which lies directly above it.

curves. These ODEs can be solved and pieced back together to find the solution of the PDE.

We assume that the curve $x(t)$ is given and study u along this curve, $u(x(t), t)$. From the chain rule, we obtain

$$\frac{d}{dt}\Big[u(x(t), t)\Big] = \frac{\partial u}{\partial x}(x(t), t) \cdot \frac{dx(t)}{dt} + \frac{\partial u}{\partial t}(x(t), t).$$

Note the clear distinction between a total derivative d/dt and a partial derivative $\partial/\partial t$. Now, if we compare the last expression with the original PDE, we see that these expressions are identical if

$$\frac{dx(t)}{dt} = c.$$

Indeed, for this choice of $x = x(t)$, we have

$$\frac{d}{dt}\Big[u(x(t), t)\Big] = c\,\frac{\partial u}{\partial x}(x(t), t) + \frac{\partial u}{\partial t}(x(t), t) = 0;$$

hence, the solution is constant along the curve $x = x(t)$. Thus, to make all this happen; that is, to apply the idea of characteristic curves, we have to solve two ODEs, which are called the **characteristic ODEs**:

$$\frac{dx(t)}{dt} = c, \tag{10.2}$$

$$\frac{d}{dt}\Big[u(x(t), t)\Big] = 0. \tag{10.3}$$

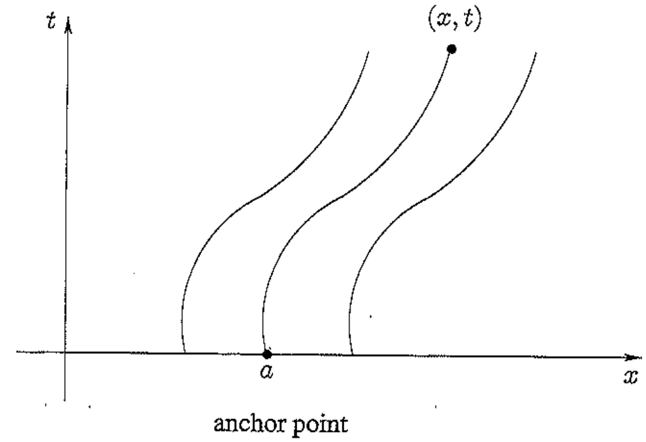


Figure 10.2 Schematic of a typical characteristic curve $x(t)$ and the anchor point. Notice that, in general, characteristic curves are not necessarily straight lines.

We solve the first of these characteristic ODEs to get

$$x(t) = ct + a,$$

where a is an unknown constant.

We observe that at $t = 0$, we have $x(0) = a$, and hence this constant a is the point where the characteristic curve starts. We call it the **anchor point** (Fig. 10.2). If a point (x, t) is given, we can always find the corresponding anchor point as

$$a = x - ct.$$

Now we solve the second characteristic ODE to get

$$u(x(t), t) = k,$$

where k is also an arbitrary constant.

If u is constant along the characteristic, it must have the same value that it has at the point where the characteristic starts, that is, at the anchor point. Hence,

$$u(x(t), t) = u(x(0), 0) = f(x(0)) = f(a) = f(x(t) - ct).$$

Here we used the initial condition $u(x, 0) = f(x)$. Therefore, given a point (x, t), there is a unique characteristic curve passing through (x, t), with anchor point at $a = x - ct$, and the solution at (x, t) is

$$u(x, t) = f(a) = f(x - ct). \tag{10.4}$$

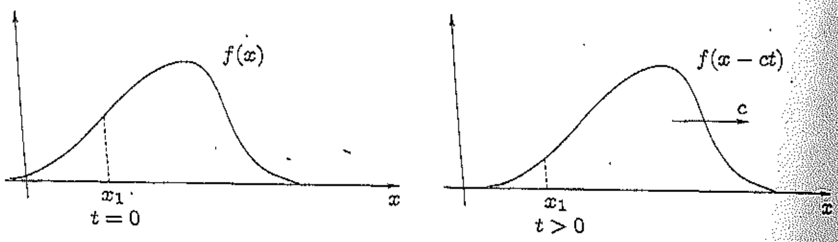


Figure 10.3 Schematic of a traveling wave that moves to the right with constant speed c.

This is a translation of the initial data to the right with velocity c, as shown in Figure 10.3.

Example 10.1. Solve the following PDE for $u(x,t)$:

$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = e^{-x^2},$$

using the method of characteristics.

Solution. The first characteristic equation is

$$\frac{dx(t)}{dt} = 5,$$

with solution

$$x(t) = 5t + a,$$

so that

$$a = x(t) - 5t.$$

The second characteristic equation has the solution

$$u(x(t),t) = f(a) = e^{-a^2} = e^{-(x(t)-5t)^2}.$$

Thus, the solution is

$$u(x,t) = e^{-(x-5t)^2}.$$

∎

Now we extend the method to allow for linear source or sink terms.

Example 10.2. Solve the following PDE for $u(x,t)$ on $-\infty < x < \infty$

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} + \beta u = 0,$$

$$u(x,0) = f(x),$$

using the method of characteristics.

Solution. Again we look for solutions of the form $u = u(x(t),t)$. From the chain rule we get

$$\frac{d}{dt}\Big[u(x(t),t)\Big] = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial t}.$$

Hence, the characteristic equations become

$$\frac{dx(t)}{dt} = \alpha,$$

$$\frac{d}{dt}\Big[u(x(t),t)\Big] = -\beta u(x(t),t).$$

The solution to the first characteristic equation is

$$x(t) = \alpha t + a.$$

Thus, the anchor point for the characteristic passing through the point (x,t) is

$$a = x - \alpha t.$$

The solution to the second characteristic equation is

$$u(x(t),t) = u(x(0),0)e^{-\beta t},$$

and the solution along the characteristic $x = x(t)$ is

$$u(x(t),t) = u(a,0)e^{-\beta t} = f\big(x(t) - \alpha t\big)e^{-\beta t}.$$

Therefore, given a point (x,t), the anchor point for the unique characteristic passing through the point is

$$a = x - \alpha t,$$

and the solution at the point (x,t) is

$$u(x,t) = f(x - \alpha t)e^{-\beta t}.$$

∎

10.2 GEOMETRIC INTERPRETATION

In Section 10.1 we considered some simple homogeneous first-order PDEs. Now we extend the previous concepts to more general PDEs of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C_1(x,y)u = C_0(x,y), \qquad (10.5)$$

where we assume that the coefficient functions are sufficiently smooth. It is convenient to rewrite this PDE in the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u), \tag{10.6}$$

where $C(x,y,u) = C_0(x,y) - C_1(x,y)u$. In applications this PDE is usually accompanied by an auxiliary condition (called an initial condition if one of the independent variables is time) of the form

$$u(x,y) = f(x,y) \tag{10.7}$$

for $(x,y) \in \Gamma_a$, where Γ_a is a curve of anchor points (Fig. 10.1) and f is a given function.

It turns out that there is a nice geometric interpretation for this PDE. Suppose that $u(x,y)$ is the solution to (10.6) subject to the auxiliary condition equation (10.7). We can think of $z = u(x,y)$ as a surface S in $\mathbb{R}^3$. Denote by Γ the curve on the surface $z = u(x,y)$ whose projection onto the xy-plane is Γ_a (Fig. 10.1). The curve Γ is called the **initial curve**. If we parameterize Γ_a using the anchor points,

$$\Gamma_a : \begin{cases} x = x_0(a), \\ y = y_0(a), \end{cases} \tag{10.8}$$

the initial curve Γ is given by

$$\Gamma : \begin{cases} x = x_0(a), \\ y = y_0(a), \\ z = z_0(a) = f(x_0(a), y_0(a)). \end{cases} \tag{10.9}$$

Recall from elementary calculus that a normal vector to the surface $z = u(x,y)$ is given by

$$\mathbf{N} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right).$$

If equation (10.6) is rewritten as

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} - C = 0,$$

this PDE says that the vector field

$$\mathbf{F} = \big(A(x,y), B(x,y), C(x,y,u)\big)$$

is perpendicular to the normal vector; that is,

$$\mathbf{F}\cdot\mathbf{N} = (A,B,C)\cdot\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right) = A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} - C = 0.$$

Therefore, the vector field $\mathbf{F}$ is everywhere tangent to the solution surface $z = u(x,y)$. In other words, the solution $u(x,y)$ to the PDE (10.6), when considered as a surface S in $\mathbb{R}^3$, is made up of the integral curves of the vector field $\mathbf{F}$.

Integral curves

$$\mathbf{r}(s) = (x(s), y(s), z(s))$$

of $\mathbf{F}$ which start from the initial curve Γ satisfy the following vector ODE:

$$\frac{d\mathbf{r}}{ds} = \mathbf{F},$$

$$\mathbf{r}(0) = (x_0(a), y_0(a), z_0(a)),$$

or, in component form, the system of ODEs

$$\frac{dx}{ds} = A(x,y), \qquad x(0) = x_0(a), \tag{10.10}$$

$$\frac{dy}{ds} = B(x,y), \qquad y(0) = y_0(a), \tag{10.11}$$

$$\frac{dz}{ds} = C(x,y,z), \qquad z(0) = z_0(a). \tag{10.12}$$

The system of ODEs (10.10), (10.11), and (10.12) are called the **characteristic equations** for the PDE given in Equation (10.6). The solutions to the characteristic equations are called the **characteristic curves** for the PDE.

Solution procedure

1. Solve the first two characteristic equations (10.10), and (10.11) to get x and y in terms of the characteristic variable s and the anchor point a:

$$x = X(s,a),$$

$$y = Y(s,a).$$

2. Insert the solution from the previous step into equation (10.12), and solve the resulting equation for z:

$$z = Z(s,a).$$

3. Write the characteristic variable s and anchor point a in terms of the original independent variables x and y; that is, invert

$$x = X(s,a),$$
$$y = Y(s,a)$$

to get

$$s = S(x,y),$$
$$a = \Lambda(x,y).$$

4. Write the solution for z in terms of x and y to get the solution to the original PDE:

$$u(x,y) = Z\big(S(x,y), \Lambda(x,y)\big).$$

Remarks on the solution procedure

1. (a) The first two characteristic equations do not involve z and so can be solved independent of the third characteristic equation. However, if $A(x,y)$ and $B(x,y)$ are complicated functions, these equations are nonlinear and can be very difficult to solve.

   (b) Normally, the solution to an ODE is a function of only one variable, the independent variable, which in this case is the characteristic variable s. But clearly, the solution also depends on the initial condition, and in this case, the initial condition depends on the anchor point a. Thus, we consider the solution of these characteristic equations as functions of both s and a.

2. Recall that $C(x,y,z) = C_0(x,y) - C_1(x,y)z$, so that equation (10.12) can be written as

$$\frac{dz}{ds} + C_1(x,y)z = C_0(x,y).$$

   This is a first-order linear ODE in z, and once x and y are known in terms of the characteristic variable s, it can be solved easily by means of an integrating factor.

3. We can think of $\{s,a\}$ and $\{x,y\}$ as two different coordinate systems for our solution surface S. Solving the characteristic equations gives us the solution surface S in terms of $\{s,a\}$; however, we would like to express our solution $z = u(x,y)$ in terms of our original independent variables x and y. This leads us to attempt to find the inverse transformation. However, the functions $X(s,a)$ and $Y(s,a)$ are often quite complicated, so that finding the inverse transformation can be rather difficult.

4. Once the first three steps have been completed, the solution is known explicitly.

5. If we apply this method to time-dependent problems of the form

$$\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} = C(x,t),$$
$$u(x,0) = f(x),$$

   we have $A(x,t) = 1$, and we can easily solve the first characteristic equation (10.10), getting $x = t$. Note that you have to make a mental shift from (y,x) to (x,t), Hence, we can chose the time t as a parameter along the characteristic curves $x(t)$; then the second and third characteristic equations (10.11) and (10.12) become

$$\frac{dx}{dt} = B(x,t),$$
$$\frac{du}{dt} = C(x,t).$$

   In this context, the two equations above are called the **characteristic equations**. Hence, the method from Section 10.1 is a simplified version of this more general geometric method.

To summarize: steps 1 and 3 can be difficult, while steps 2 and 4 are easy. We illustrate the solution procedure outlined above with several examples.

Example 10.3. Solve the following PDE:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x),$$

where c is a constant.

Solution. This is the example we solved at the beginning of the chapter. We want to apply the solution method outlined above, but in this example the independent variables are $\{x,t\}$ rather than $\{x,y\}$. From the auxiliary condition, in this case the initial condition $u(x,0) = f(x)$, we see that the curve of anchor points, Γ_a, is just the x-axis. Thus,

$$\Gamma_a = \{\, (x,t) \mid x \in \mathbb{R},\ t = 0 \,\},$$

which can be parameterized as follows:

$$\Gamma_a : \begin{cases} x = x_0(a) = a, \\ t = t_0(a) = 0 \end{cases}$$

for $-\infty < a < \infty$.

This leads us to the following parametric representation of the initial curve Γ:

$$\Gamma : \begin{cases} x = x_0(a) = a, \\ t = t_0(a) = 0, \\ z = z_0(a) = f(a), \end{cases}$$

for $-\infty < a < \infty$. For this example the characteristic equations are

$$\frac{dx}{ds} = c, \qquad x(0) = a, \tag{10.13}$$

$$\frac{dt}{ds} = 1, \qquad t(0) = 0, \tag{10.14}$$

$$\frac{dz}{ds} = 0, \qquad z(0) = f(a). \tag{10.15}$$

We now proceed to solve the problem.

Step 1: Solve the first two characteristic equations.
In this case these ODEs are easily solved by direct integration, yielding

$$x = X(s,a) = cs + a,$$
$$t = T(s,a) = s.$$

Step 2: Solve the third characteristic equation.
Again, this is easily done, giving

$$z = Z(s,a) = f(a).$$

Step 3: Invert the transformation.
We invert

$$x = cs + a,$$
$$t = s,$$

to get

$$s = S(x,t) = t,$$
$$a = \Lambda(x,t) = x - ct.$$

Step 4: Write down the final solution.

$$u(x,t) = Z(S(x,t), \Lambda(x,t)) = f(\Lambda(x,t)) = f(x - ct),$$

which is the same solution as found earlier: namely, equation (10.4).

Example 10.4. Solve the following PDE:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u, \quad -\infty < x < \infty, \quad y > 0,$$
$$u(x, x^2) = 1 + x^2$$

using the method of characteristics.

Solution. In this problem, the side condition is not an initial condition at $x = 0$ but, rather, a condition along the parabola $y = x^2$. Hence, the curve of anchor points Γ_a and the initial curve Γ can be parameterized as follows:

$$\Gamma_a : \begin{cases} x = a, \\ y = a^2 \end{cases}$$

for $-\infty < a < \infty$, and

$$\Gamma : \begin{cases} x = a, \\ y = a^2, \\ z = 1 + a^2 \end{cases}$$

for $-\infty < a < \infty$. For this example the characteristic equations are

$$\frac{dx}{ds} = x, \tag{10.16}$$
$$x(0) = a, \tag{10.17}$$
$$\frac{dy}{ds} = y, \tag{10.18}$$
$$y(0) = a^2, \tag{10.19}$$
$$\frac{dz}{ds} = z, \tag{10.20}$$
$$z(0) = 1 + a^2. \tag{10.21}$$

Step 1: The first two characteristic equations can be integrated to obtain

$$x = X(s,a) = ae^s,$$
$$y = Y(s,a) = a^2e^s.$$

Step 2: The third characteristic equation can be solved to obtain

$$z = Z(s,a) = (1+a^2)e^s.$$

Step 3: We can invert the transformation

$$x = ae^s,$$
$$y = a^2e^s,$$

to obtain

$$s = S(x,y) = \log\frac{x^2}{y},$$
$$a = \Lambda(x,y) = \frac{y}{x}.$$

Step 4: We write the final solution as

$$u(x,y) = Z\left(\log\frac{x^2}{y}, \frac{y}{x}\right) = \left(1+\left(\frac{y}{x}\right)^2\right)\frac{x^2}{y},$$

that is,

$$u(x,y) = \frac{x^2+y^2}{y}.$$

Example 10.5. Solve the following PDE for $u(x,y)$:

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = y, \quad x>0, \quad -\infty<y<\infty,$$
$$u(0,y) = 1+y^2$$

using the method of characteristics.

Solution. From the auxiliary condition we see that the curve of anchor points, Γ_a, is just the y-axis, that is, the line $x=0$. Thus, the curves Γ_a and Γ can be parameterized as follows:

$$\Gamma_a : \begin{cases} x = 0, \\ y = a \end{cases}$$

and

$$\Gamma : \begin{cases} x = 0, \\ y = a, \\ z = 1+a^2 \end{cases}$$

for $-\infty < a < \infty$. For this example the characteristic equations are

$$\frac{dx}{ds} = 1, \tag{10.22}$$
$$x(0) = 0, \tag{10.23}$$
$$\frac{dy}{ds} = 2x, \tag{10.24}$$
$$y(0) = a, \tag{10.25}$$
$$\frac{dz}{ds} = y, \tag{10.26}$$
$$z(0) = 1+a^2. \tag{10.27}$$

We now proceed to solve the problem.

Step 1: Solve the first two characteristic equations.

The first characteristic equation can be integrated to obtain $x = s$. Inserting this into the second characteristic equation yields

$$\frac{dy}{ds} = 2s.$$

Integrating this equation and applying the initial condition results in

$$y = s^2 + a.$$

Thus,

$$x = X(s,a) = s,$$
$$y = Y(s,a) = s^2 + a.$$

Step 2: Solve the third characteristic equation.

Using the results of step 1, the third characteristic equation becomes

$$\frac{dz}{ds} = s^2 + a,$$

and integration yields

$$z = \frac{s^3}{3} + as + c,$$

where c is a constant of integration. The initial condition implies that

$$c = 1 + a^2,$$

and the result is

$$z = Z(s, a) = \frac{s^3}{3} + as + 1 + a^2.$$

Step 3: Invert the transformation.

We invert

$$x = s,$$

$$y = s^2 + a$$

to get

$$s = x,$$

$$a = y - x^2.$$

Step 4: Write the final solution.

The solution is

$$u(x, y) = Z(x, y - x^2) = \frac{x^3}{3} + (y - x^2)x + 1 + (y - x^2)^2;$$

that is,

$$u(x, y) = x^4 - \frac{2}{3}x^3 - 2x^2 y + xy + y^2 + 1.$$

10.3 D'ALEMBERT'S SOLUTION

The method of characteristics can also be used to derive d'Alembert's solution to the one-dimensional wave equation on the entire real line $-\infty < x < \infty$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = f(x),$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x).$$

We assume that the solution is twice continuously differentiable and is such that

$$\frac{\partial^2 u}{\partial x\, \partial t} = \frac{\partial^2 u}{\partial t\, \partial x}.$$

We take a more abstract view and perform algebraic manipulations of the differential operators. We use the factorization of the binomial $a^2 - b^2$ as

$$a^2 - b^2 = (a + b)(a - b) = (a - b)(a + b)$$

to factor the wave operator

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}.$$

We can write the wave equation as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0. \quad (10.28)$$

Now we introduce a new variable:

$$v(x, t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x, t) = \frac{\partial u}{\partial t}(x, t) - c\frac{\partial u}{\partial x}(x, t).$$

At $t = 0$ we have

$$v(x, 0) = \frac{\partial u}{\partial t}(x, 0) - c\frac{\partial u}{\partial x}(x, 0) = g(x) - cf'(x).$$

We can rewrite the second-order wave equation as a pair of first-order linear PDEs to be solved sequentially:

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0, \quad (10.29)$$

$$v(x, 0) = F(x)$$

and

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v, \quad (10.30)$$

$$u(x, 0) = f(x),$$

where $F(x) = g(x) - c\,f'(x)$. The first of these is, in fact, the simple first-order equation (10.1), which we studied at the beginning of this chapter, and its solution had the form

$$v(x, t) = F(x - ct). \quad (10.31)$$

Now we put this solution for v into equation (10.30) to get

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = F(x - ct), \qquad (10.32)$$
$$u(x,0) = f(x).$$

where $F(x) = g(x) - c\,f'(x)$, and we solve problem (10.32) using the method of characteristics.

From the auxiliary condition $u(x,0) = f(x)$, we see that the curve of anchor points, Γ_a, is just the x-axis, as was the case in the example at the beginning of this chapter. Thus, the curve Γ_a and the initial curve Γ can be parameterized as follows:

$$\Gamma_a : \begin{cases} x = x_0(a) = a, \\ t = t_0(a) = 0 \end{cases}$$

and

$$\Gamma : \begin{cases} x = x_0(a) = a, \\ t = t_0(a) = 0, \\ z = z_0(a) = f(a) \end{cases}$$

for $-\infty < a < \infty$.

Since the coefficient $A(x,t) = 1$, we can use the simpler approach mentioned in the *Remarks on the solution procedure* and choose $s = t$ as the parameter. In this case the remaining characteristic equations are

$$\frac{dx}{dt} = -c, \qquad (10.33)$$
$$x(0) = a, \qquad (10.34)$$
$$\frac{dz}{dt} = F(x - ct), \qquad (10.35)$$
$$z(0) = f(a). \qquad (10.36)$$

We now proceed to solve the problem.

Step 1: Solve the first two characteristic equations.

In this case this ODE is easily solved by direct integration, yielding

$$x = X(t,a) = -ct + a.$$

Step 2: Solve the third characteristic equation.

Using the results of step 1, the third characteristic equation becomes

$$\frac{dz}{dt} = F(-ct - a - ct) = F(a - 2ct),$$

and integration yields

$$z = \int_0^t F(a - 2c\xi)\,d\xi + K,$$

where K is a constant of integration. The initial condition implies that $K = f(a)$, so we obtain

$$z = Z(t,a) = \int_0^t F(a - 2c\xi)\,d\xi + f(a),$$

and recalling the definition of F in terms of f and g, $F(x) = g(x) - c\,f'(x)$, and making the substitution $\mu = a - 2c\,\xi$, the integral of F becomes

$$\begin{aligned}\int_0^t F(x - 2c\,\xi)\,d\xi &= -\frac{1}{2c}\int_a^{a-2ct} F(\mu)\,d\mu \\ &= -\frac{1}{2c}\int_a^{a-2ct} [g(\mu) - cf'(\mu)]\,d\mu \\ &= -\frac{1}{2c}\int_a^{a-2ct} g(\mu)\,d\mu + \frac{1}{2}f(\mu)\Big|_a^{a-2ct} \\ &= \frac{1}{2c}\int_{a-2ct}^{a} g(\mu)\,d\mu + \frac{1}{2}\big[f(a-2ct) - f(a)\big].\end{aligned}$$

The final result is

$$Z(t,a) = \frac{1}{2}\big[f(a) + f(a-2ct)\big] + \frac{1}{2c}\int_{a-2ct}^{a} g(\mu)\,d\mu.$$

Step 3: Invert the transformation.

We invert $x = -ct + a$ to get $a = \Lambda(x,t) = x + ct$.

Step 4: Write the final solution.

We have

$$\begin{aligned}u(x,t) &= Z\big(t, \Lambda(x,t)\big) = Z(t, x+ct) \\ &= \frac{1}{2}[f(x+ct) + f(x+ct-2ct)] + \frac{1}{2c}\int_{x+ct-2ct}^{x+ct} g(\mu)\,d\mu,\end{aligned}$$

and simplifying this expression, we recover d'Alembert's solution:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\,d\mu.$$

10.4 EXTENSION TO QUASILINEAR EQUATIONS

Up to this point we have considered only linear equations. However, the equations often encountered in applications are, at least mildly, nonlinear. The extension to quasilinear PDEs which are nonlinear in u, but still linear in u_x and u_y is quite straightforward. We rewrite equation (10.6) as

$$A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u), \tag{10.37}$$

but now allow the coefficient functions A and B to depend on the unknown function u in addition to the independent variables x and y. In addition, we no longer restrict C to be linear in u.

The characteristic equations are

$$\frac{dx}{ds} = A(x,y,z), \tag{10.38}$$
$$x(0) = x_0(a), \tag{10.39}$$
$$\frac{dy}{ds} = B(x,y,z), \tag{10.40}$$
$$y(0) = y_0(a), \tag{10.41}$$
$$\frac{dz}{ds} = C(x,y,z), \tag{10.42}$$
$$z(0) = z_0(a). \tag{10.43}$$

What is different here is that the first two characteristic equations are no longer independent of z, and now the complete system must be solved. However, once this is done, we proceed with steps 3 and 4 as before to obtain the solution.

Example 10.6. (*A Quasilinear Example*)
Solve the first-order quasilinear equation

$$(y-u)\frac{\partial u}{\partial x} + (u-x)\frac{\partial u}{\partial y} = x - y, \quad -\infty < x, y < \infty,$$
$$u\left(x, \frac{1}{x}\right) = 0.$$

Solution. The side condition is given along the hyperbola

$$\Gamma_a = \left\{ \left(a, \frac{1}{a}\right) \,\middle|\, a \neq 0 \right\}.$$

The characteristic equations are

$$\frac{dx}{ds} = y - z,$$
$$\frac{dy}{ds} = z - x,$$
$$\frac{dz}{ds} = x - y,$$

and since all equations are coupled, they have to be solved simultaneously.

Under closer inspection, we find two conserved quantities:

- First,

$$\frac{d}{ds}\Big(x(s)+y(s)+z(s)\Big) = \frac{dx}{ds}(s) + \frac{dy}{ds}(s) + \frac{dz}{ds}(s)$$
$$= \Big(y(s)-z(s)\Big) + \Big(z(s)-x(s)\Big) + \Big(x(s)-y(s)\Big)$$
$$= 0,$$

and hence there is a constant A such that

$$x(s)+y(s)+z(s) = A$$

along the characteristics.

- Next,

$$\frac{d}{ds}\Big(x(s)^2+y(s)^2+z(s)^2\Big) = 2\Big(x(s)x'(s)+y(s)y'(s)+z(s)z'(s)\Big)$$
$$= 2\Big(x(s)\big[y(s)-z(s)\big] + y(s)\big[z(s)-x(s)\big] + z(s)\big[x(s)-y(s)\big]\Big)$$
$$= 0,$$

and hence there is a constant B such that

$$x(s)^2+y(s)^2+z(s)^2 = B$$

along the characteristics.

All we know so far is that the characteristics lie on the intersection of a plane and a sphere in $\mathbb{R}^3$, so they are circles, or points. We still cannot solve explicitly for $x(s)$, $y(s)$, and $z(s)$, but we can use the side conditions to compute the unknown constants A and B as follows.

On Γ_a we have $u(a, 1/a) = 0$; hence,

$$a + \frac{1}{a} + 0 = \frac{a^2+1}{a} = A$$

and

$$a^2 + \frac{1}{a^2} + 0^2 = \frac{a^4+1}{a^2} = B,$$

so that

$$A^2 = \frac{a^4 + 2a^2 + 1}{a^2} = B + 2.$$

We can use this relation directly, whereby we circumvent finding explicit expressions for s and $\dot{a}$. Since $A^2 = B + 2$, then

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2,$$

and therefore

$$xy + xz + yz = 1.$$

Thus, solving for $z = u(x, y)$, we have

$$u(x, y) = \frac{1 - xy}{x + y},$$

the solution to our problem. ∎

10.5 SUMMARY

The **method of characteristics** is quite simple, and hence this chapter could have come much earlier in the book. However, herein, this method applies primarily to first-order equations. Thus, it stands alone, separate from all the other chapters. Nevertheless, using the method of characteristics, we were able to obtain a third derivation of **d'Alembert's formula** for the solution to the boundary value–initial value problem for the one-dimensional wave equation by factoring the wave operator into a product of two linear first-order differential operators.

The important idea in the method is to find characteristic curves $x(s)$, such that the PDE becomes an ODE along those curves. Once those curves $x(s)$ are found, we need to find the anchor point a, which defines the initial condition for the ODE on the characteristic curve. Finally, we solve this ODE, using the anchor point for the initial condition.

This method needs to be practiced; hence, we recommend that you do as many of the following problems as possible.

10.5.1 Problems and Notes

Problems from Part II:

Exercise	17.1	17.2	17.3	17.4	17.5	17.6
Notes						
Exercise	17.7	17.8	17.9	17.10	17.11	17.12
Notes						
Exercise	17.13	14.6	14.7	14.10		
Notes						

Final exam questions:

Exercise	19.8
Notes	

You should now be able to do **Final Exam 2**.

CHAPTER 17

METHOD OF CHARACTERISTICS PROBLEMS

Exercise 17.1. ✗✗
Assume that $u(x,t)$ is the linear density of particulate matter being carried by the wind from a dump truck at the oil sands at position $x = 0$ and time t. The wind is moving in the positive x-direction with a constant speed of k meters/sec, and the particulates are condensing out of the air at a rate $ru(x,t)$, where $r > 0$ is constant. The density u satisfies the boundary value–initial value problem

$$\frac{\partial u}{\partial t}(x,t) + k\frac{\partial u}{\partial x}(x,t) = -ru(x,t), \quad 0 < x < \infty, \quad t > 0, \tag{17.1}$$

$$u(x,0) = \phi(x),$$

where $\phi(x)$ is the initial distribution of the particle density. Solve this initial value–boundary value problem using the method of characteristics.

Solution. The method of characteristics reduces the partial differential equation to a pair of ordinary differential equations, one of which is solved for the characteristic curves in the xt-plane along which the solutions to the other equation are easily found.

Partial Differential Equations: Theory and Completely Solved Problems, First Edition. By T. Hillen, I. E. Leonard, H. van Roessel

We write the partial differential equation so that the partial differential operator resembles a directional deriative or a total derivative. For example,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = -ru, \tag{17.2}$$

where

$$\frac{dx}{dt} = k.$$

The family of curves with differential equation

$$\frac{dx}{dt} = k$$

are the **characteristic curves** of the partial differential equation (17.1). If $x = x(t)$ is a characteristic curve of (17.1), then along this curve equation (17.2) becomes

$$\frac{\partial}{\partial t}\big(u(x(t),t)\big) + \frac{\partial}{\partial x}\big(u(x(t),t)\big)\cdot\frac{dx}{dt} = -ru(x(t),t);$$

that is,

$$\frac{d}{dt}\big(u(x(t),t)\big) = -ru(x(t),t),$$

which is an ordinary differential equation for $u(x(t),t)$. Letting $v(t) = u(x(t),t)$ for $t > 0$, v satisfies the ordinary differential equation

$$\frac{dv}{dt} + rv = 0,$$

that is, a first-order linear homogeneous ordinary differential equation, and to solve it we multiply by the integrating factor $M(t) = e^{rt}$, to get

$$\frac{d}{dt}\left(e^{rt}v\right) = e^{rt}\frac{dv}{dt} + re^{rt}v = e^{rt}\left(\frac{dv}{dt} + rv\right) = 0$$

for all $t > 0$. Therefore, $e^{rt}v(t)$ is a constant, so that

$$e^{rt}v(t) = e^{r0}v(0) = v(0),$$

that is,

$$e^{rt}u(x(t),t) = u(x(0),0) = \phi(x(0)),$$

so that

$$u(x(t),t) = e^{-rt}\phi(x(0)) \tag{17.3}$$

for all $t > 0$.

Given a point (x,t) in the xt-plane with $t > 0$, there is exactly one characteristic curve that passes through this point, namely,

$$x(t) = kt + x(0),$$

where

$$x = x(t) = kt + x(0),$$

and for this characteristic curve, $x(0)$ is called the **anchor point** of the characteristic through (x,t). Therefore,

$$u(x,t) = e^{-rt}\phi(x(0)) = e^{-rt}\phi(x - kt),$$

and, since (x,t) was arbitrary, the solution to the initial value problem is given by

$$u(x,t) = e^{-rt}\phi(x - kt) \tag{17.4}$$

for $0 < x < \infty$, $t > 0$. Finally, we note that $e^{rt}u(x,t)$ is constant along the characteristic curves.

Exercise 17.2. ✗✗
Use the method of characteristics to solve the initial value problem

$$\frac{\partial w}{\partial t} + 5\frac{\partial w}{\partial x} = e^{3t}, \quad -\infty < x < \infty, \quad t > 0,$$
$$w(x,0) = e^{-x^2}.$$

Solution: Let

$$\frac{dx}{dt} = 5;$$

then along the characteristic curve $x(t) = 5t + a$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = e^{3t},$$

so that

$$w(x(t),t) = \frac{1}{3}e^{3t} + K,$$

where K is a constant, and $K = w(x(0),0) - \frac{1}{3}$, so that

$$w(x(t),t) = \frac{1}{3}e^{3t} + w(x(0),0) - \frac{1}{3} = \frac{1}{3}e^{3t} + w(a,0) - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3}.$$

Given the point (x,t), let $x = 5t + a$ be the unique characteristic curve passing through this point; then the anchor point is $a = x - 5t$ and the solution is

$$w(x,t) = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-(x-5t)^2} - \frac{1}{3}$$

for $-\infty < x < \infty$, $t > 0$.

Exercise 17.3.
Use the method of characteristics to solve the initial value problem

$$\frac{\partial w}{\partial t} - x\frac{\partial w}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$w(x,0) = x^3 - 1.$$

XX

Solution. Let

$$\frac{dx}{dt} = -x;$$

then along the characteristic curve $x(t) = x_0e^{-t}$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = 0,$$

so that

$$w(x(t),t) = K,$$

where K is a constant, and $K = w(x(0),0)$, so that

$$w(x(t),t) = w(x(0),0) = w(x_0,0) = x_0^3 - 1.$$

Given the point (x,t), let $x = x_0e^{-t}$ be the unique characteristic curve passing through this point; then $x_0 = xe^t$ is the anchor point and the solution is

$$w(x,t) = x_0^3 - 1 = x^3e^{3t} - 1$$

for $-\infty < x < \infty$, $t > 0$.

Exercise 17.4.
Use the method of characteristics to solve the initial value problem

$$\frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} = \sin 2\pi t, \quad -\infty < x < \infty, \quad t > 0,$$
$$z(x,0) = \cos x.$$

XX

Solution. Let

$$\frac{dx}{dt} = 3;$$

then along the characteristic curve $x(t) = 3t + a$, the partial differential equation becomes

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dx}{dt} = \sin 2\pi t,$$

so that

$$z(x(t),t) = -\frac{1}{2\pi}\cos 2\pi t + K,$$

where K is a constant, and

$$z(x(0),0) = -\frac{1}{2\pi} + K = \cos x(0) = \cos a = \cos(x(t) - 3t),$$

so that

$$K = \cos(x(t) - 3t) + \frac{1}{2\pi}.$$

Given the point (x,t), let $x = 3t + a$ be the unique characteristic curve passing through this point; then the anchor point is $a = x - 3t$ and the solution is

$$z(x,t) = -\frac{1}{2\pi}\cos 2\pi t + \cos(x - 3t) + \frac{1}{2\pi} \tag{17.5}$$

for $-\infty < x < \infty$ and $t > 0$.

As a check, we note that for

$$z(x,t) = -\frac{1}{2\pi}\cos 2\pi t + \cos(x - 3t) + \frac{1}{2\pi},$$

we have

$$\frac{\partial z}{\partial t} = \sin 2\pi t + 3\sin(x - 3t)$$

and

$$\frac{\partial z}{\partial x} = -\sin(x - 3t),$$

so that

$$\frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} = \sin 2\pi t.$$

Also,

$$z(x,0) = -\frac{1}{2\pi} + \cos x + \frac{1}{2\pi} = \cos x,$$

and (17.5) is a solution to the initial value problem.

Exercise 17.5.
Solve the first-order equation

$$\frac{\partial u}{\partial t} + 3x\frac{\partial u}{\partial x} = 2t, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \log(1 + x^2).$$

XX

Solution. Let
$$\frac{dx}{dt} = 3x;$$
then along the characteristic curve $x(t) = ae^{3t}$, the partial differential equation becomes
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 2t,$$
so that
$$u(x(t),t) = t^2 + K,$$
where K is a constant, and $K = u(x(0),0)$, so that
$$u(x(t),t) = t^2 + u(x(0),0) = t^2 + u(a,0) = t^2 + \log(1+a^2).$$
Given the point (x,t), let $x = ae^{3t}$ be the unique characteristic curve passing through this point; then the anchor point is $a = xe^{-3t}$ and the solution is
$$u(x,t) = t^2 + \log\left(1 + x^2 e^{-6t}\right)$$
for $-\infty < x < \infty$ and $t > 0$.

Exercise 17.6. ✗✗
Using the method of characteristics, solve
$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t > 0,$$
$$w(x,0) = f(x).$$

Solution. Let
$$\frac{dx}{dt} = c;$$
then along the characteristic curve $x(t) = ct + a$, the partial differential equation becomes
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = e^{2x(t)} = e^{2(ct+a)},$$
so that
$$w(x(t),t) = \frac{1}{2c}e^{2ct+2a} + K = \frac{1}{2c}e^{2x(t)} + K,$$
where K is a constant, and $K = w(x(0),0) - (1/2c)e^{2x(0)}$, so that
$$w(x(t),t) = \frac{1}{2c}e^{2x(t)} + f(x(0)) - \frac{1}{2c}e^{2x(0)};$$
that is,
$$w(x(t),t) = \frac{1}{2c}e^{2x(t)} + f(x(t) - ct) - \frac{1}{2c}e^{2(x(t)-ct)}.$$

Given the point (x,t), let $x = ct + a$ be the unique characteristic curve passing through this point, then the anchor point is $a = x - ct$ and the solution is
$$w(x,t) = \frac{1}{2c}e^{2x}\left(1 - e^{-2ct}\right) + f(x - ct)$$
for $-\infty < x < \infty$ and $t > 0$.

Exercise 17.7. ✗✗
Using the method of characteristics, solve
$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1, \quad -\infty < x < \infty, \quad t > 0,$$
$$w(x,0) = f(x).$$

Solution. Let
$$\frac{dx}{dt} = t;$$
then along the characteristic curve $x(t) = (t^2/2) + a$, the partial differential equation becomes
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = 1,$$
so that
$$w(x(t),t) = t + K,$$
where K is a constant, and $K = w(x(0),0)$, so that
$$w(x(t),t) = t + w(x(0),0) = t + f(a).$$
Given the point (x,t), let $x = (t^2/2) + a$ be the unique characteristic curve passing through this point; then the anchor point is $a = x - (t^2/2)$ and the solution is
$$w(x,t) = t + f\left(x - \frac{t^2}{2}\right)$$
for $-\infty < x < \infty$ and $t > 0$.

Exercise 17.8. ✗✗
Consider
$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x).$$
Show that the characteristics are straight lines.

Solution. Along the characteristic curve $x = x(t)$ whose differential equation is

$$\frac{dx}{dt} = 2u(x(t), t),$$

the partial differential equation becomes

$$\frac{d}{dt}\left[u(x(t), t)\right] = \frac{\partial u}{\partial t}(x(t), t) + \frac{dx}{dt} \cdot \frac{\partial u}{\partial x}(x(t), t) = 0,$$

so that $u(x(t), t) = \text{ constant } = u(x(0), 0)$, and

$$\frac{dx}{dt} = 2u(x(t), t) = 2u(x(0), 0),$$

so that

$$x(t) = 2u(x(0), 0)t + x(0) = 2f(x(0))t + x(0)$$

for $t > 0$, and the characteristic curves are the straight lines $x = 2f(x_0)t + x_0$ and intersect the x-axis at the point x_0.

Exercise 17.9. ✗✗✗
Consider

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

with

$$u(x, 0) = f(x) = \begin{cases} 1, & x < 0, \\ 1 + x/a, & 0 < x < a, \\ 2, & x > a. \end{cases}$$

(a) Determine the equations for the characteristics. Sketch the characteristics.

(b) Determine the solution $u(x, t)$. Sketch $u(x, t)$ for t fixed.

Solution.

(a) The equations for the characteristics are $x = 2f(x_0)t + x_0$, where the parameter x_0 is the intersection of the characteristic with the x-axis for $-\infty < x_0 < \infty$.

(i) For $x_0 < 0$, we have $f(x_0) = 1$, and the characteristics have the equation $x = 2t + x_0$.

(ii) For $0 < x_0 < a$, we have $f(x_0) = 1 + x_0/a$, and the characteristics have the equation $x = 2(1 + x_0/a)t + x_0$.

(iii) For $x > a$, we have $f(x_0) = 2$, and the characteristics have the equation $x = 4t + x_0$.

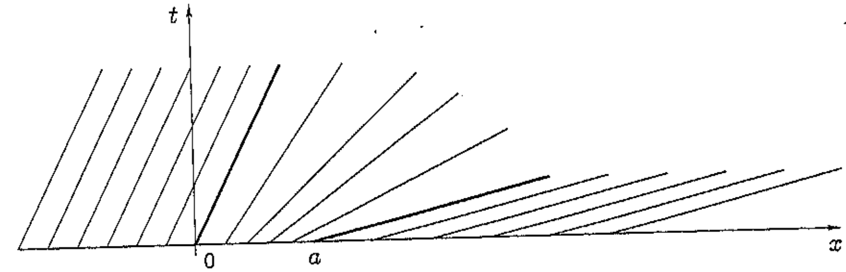


Figure 17.1 Characteristic fan.

The characteristics are sketched in Figure 17.1.

(b) The solution along the characteristic $x = 2f(x_0)t + x_0$ is given by

$$u(x, t) = f(x_0),$$

and considering the cases where

$$x_0 < 0, \qquad 0 < x_0 < a, \qquad \text{and} \qquad a < x_0,$$

we have

$$u(x, t) = \begin{cases} 1, & \text{for } x < 2t, \\ \dfrac{x + a}{a + 2t}, & \text{for } 2t < x < 4t + a, \\ 2, & \text{for } x > 4t + a. \end{cases}$$

For a fixed $t > 0$, the solution is sketched in Figure 17.2.

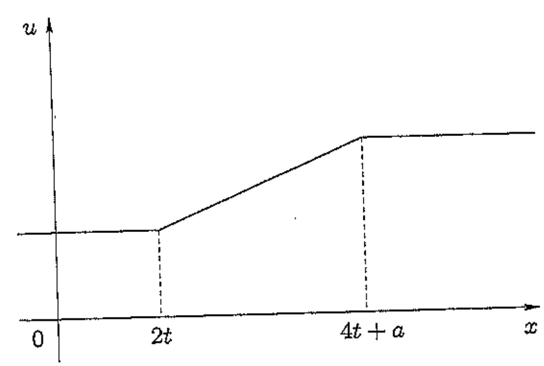


Figure 17.2 Graph of the solution.

Exercise 17.10. XX
Derive the general solution of the equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = u, \quad a,\, b \neq 0$$

by using the change of variables $\alpha = Ax + Bt$ and $\beta = Cx + Dt$, where A, B, C, and D are to be determined so as to reduce the partial differential equation to an ordinary differential equation, which we can then solve.

Solution. Let

$$\alpha = Ax + Bt \qquad \text{and} \qquad \beta = Cx + Dt;$$

from the chain rule, we have

$$\frac{\partial u}{\partial x} = A\frac{\partial u}{\partial \alpha} + C\frac{\partial u}{\partial \beta}$$
$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial \alpha} + D\frac{\partial u}{\partial \beta},$$

and the original partial differential equation becomes

$$(aB + bA)\frac{\partial u}{\partial \alpha} + (aD + bC)\frac{\partial u}{\partial \beta} = u.$$

Now let $B = -b$, $A = a$, $C = 0$, and $D = 1/a$; then the equation becomes

$$\frac{\partial u}{\partial \beta} - u = 0,$$

and multiplying this equation by $e^{-\beta}$, we have

$$e^{-\beta}\frac{\partial u}{\partial \beta} - e^{-\beta}u = 0;$$

that is,

$$\frac{\partial}{\partial \beta}\left(e^{-\beta}u\right) = 0,$$

and the quantity $e^{-\beta}u$ is independent of β. Therefore, the solution is

$$u(\alpha, \beta) = f(\alpha)e^{\beta},$$

where f is an arbitrary function of α. In terms of the original variables, the solution is

$$u(x,t) = f(ax - bt)e^{t/a}.$$

Exercise 17.11. XX
Solve the following partial differential equation on an infinite domain:

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 2t + \sin t, \quad -\infty < x < \infty, \quad t > 0,$$
$$w(x,0) = x^2 + 5.$$

At the end, you might check that your solution really satisfies this problem.

Solution. Let $dx/dt = t$, then along the characteristic curve $x(t) = (t^2/2) + a$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = 2t + \sin t,$$

so that

$$w(x(t),t) = t^2 - \cos t + K,$$

where K is a constant, and $K = w(x(0),0) + 1$, and

$$w(x(t),t) = t^2 - \cos t + w(x(0),0) + 1 = t^2 - \cos t + w(a,0) + 1$$
$$= t^2 - \cos t + a^2 + 5 + 1 = t^2 - \cos t + \left(x(t) - \frac{t^2}{2}\right)^2 + 6.$$

Given a point (x,t) in the xt-plane, let $x = (t^2/2) + a$ be the unique characteristic curve passing through this point; then the anchor point is $a = x - (t^2/2)$ and the solution is

$$w(x,t) = t^2 - \cos t + a^2 + 6 = t^2 - \cos t + \left(x - \frac{t^2}{2}\right)^2 + 6 \qquad (17.6)$$

for $-\infty < x < \infty$, $t > 0$.
As a check, note that for this function $w(x,t)$ we have

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 2t + \sin t + 2\left(x - \frac{t^2}{2}\right)(-t) + 2t\left(x - \frac{t^2}{2}\right) = 2t + \sin t$$

and it satisfies the partial differential equation, while

$$w(x,0) = x^2 + 5,$$

and (17.6) is the solution to this problem.

Exercise 17.12. XXXX

The displacement $u = u(x,t)$ of an infinitely long string is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0.$$

At time $t = 0$ an initial signal is given of the form

$$u(x,0) = f(x) = \begin{cases} x, & \text{for } 0 < x < 1, \\ -x+2, & \text{for } 1 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad -\infty < x < \infty.$$

(a) Solve this problem.

(b) Sketch the solution for times t_1, t_2, t_3, t_4, t_5, with

$$t_1 = 0, \quad 0 < t_2 < 1/4, \quad t_3 = 1/4, \quad 1/4 < t_4 < 1/2, \quad t_5 = 1/2.$$

(c) At what time does the signal reach the point $\overline{x} = 11$?

Solution.

(a) d'Alembert's solution to the wave equation is given by

$$u(x,t) = \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} g(s)\,ds,$$

where

$$f(x+2t) = \begin{cases} x+2t, & \text{if } 0 < x+2t < 1, \\ -x-2t+2, & \text{if } 1 < x+2t < 2, \\ 0, & \text{if } x+2t \leqslant 0 \text{ or } x+2t \geqslant 2 \end{cases}$$

and

$$f(x-2t) = \begin{cases} x-2t, & \text{if } 0 < x-2t < 1, \\ -x+2t+2, & \text{if } 1 < x-2t \leqslant 2, \\ 0, & \text{if } x-2t \leqslant 0 \text{ or } x-2t \geqslant 2, \end{cases}$$

and $g(x) = 0$ for $-\infty < x < \infty$.

(b) In Figure 17.3 we sketch the solution by considering 10 regions in the xt-plane bounded by forward-facing and backward-facing characteristics. First note that $u = 0$ in regions I, IV, and VII.

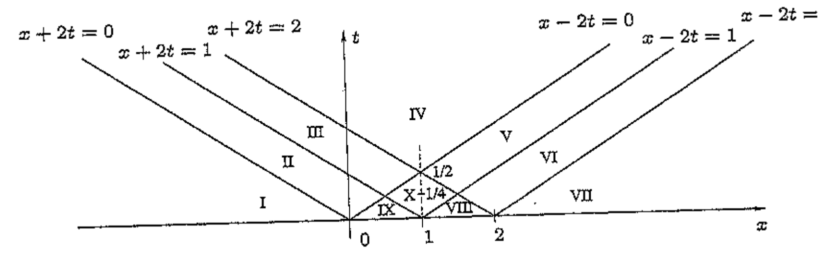


Figure 17.3

Region II: $0 < x + 2t < 1$ and $x - 2t < 0$; that is,

$$-2t < x < 1 - 2t \quad \text{and} \quad x < 2t,$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}(x + 2t + 0) = \frac{1}{2}(x + 2t).$$

Region III: $1 < x + 2t < 2$ and $x - 2t < 0$; that is,

$$1 - 2t < x < 2 - 2t \quad \text{and} \quad x < 2t,$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}(-x - 2t + 2 + 0) = \frac{1}{2}(2 - x - 2t).$$

Region V: $0 < x - 2t < 1$ and $x + 2t > 2$; that is,

$$2t < x < 1 + 2t \quad \text{and} \quad x > 2 - 2t,$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}(x - 2t + 0) = \frac{1}{2}(x - 2t).$$

Region VI: $1 < x - 2t < 2$ and $x + 2t > 2$; that is,

$$1 + 2t < x < 2 + 2t \quad \text{and} \quad x > 2 - 2t,$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}(-x + 2t + 2 + 0) = \frac{1}{2}(2 - x + 2t).$$

Region VIII: $1 < x - 2t < 2$ and 1

$$1 + 2t < x < 2 + 2t \qquad \text{and}$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left(-x + 2t + 2 - x - 2t + \right.$$

Region IX: $0 < x - 2t < 1$ and $0 < x + 2t < 1$; that is,

$$2t < x < 1 + 2t \qquad \text{and} \qquad -2t < x < 1 - 2t,$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}(x + 2t + x - 2t) = x.$$

Region X: $0 < x - 2t < 1$ and $1 < x + 2t < 2$; that is,

$$2t < x < 1 + 2t \qquad \text{and} \qquad 1 - 2t < x < 2 - 2t,$$

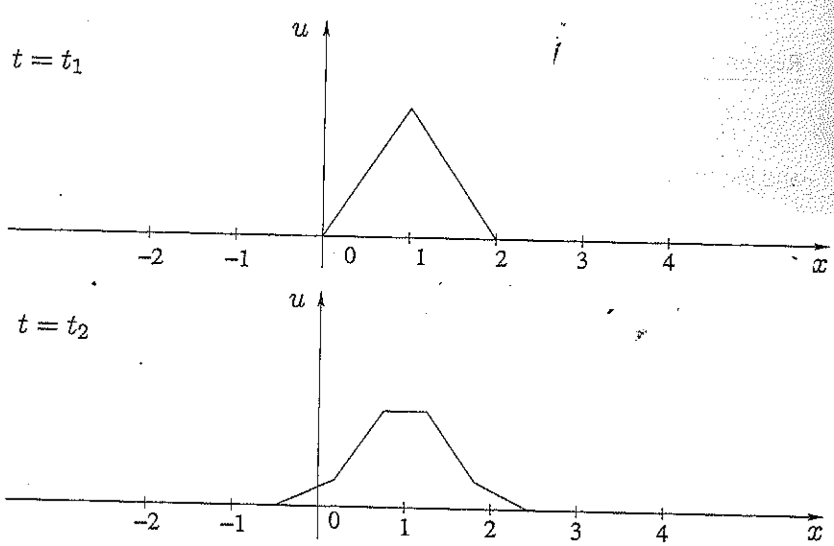
and the solution in this region is

$$u(x,t) = \frac{1}{2}(x - 2t - x - 2t + 2) = 1 - 2t.$$

We sketch the solution for

$$t_1 = 0, \qquad 0 < t_2 < \frac{1}{4}, \qquad t_3 = \frac{1}{4}, \qquad \frac{1}{4} < t_4 < \frac{1}{2}, \qquad t_5 = \frac{1}{2}$$

in Figure 17.4.



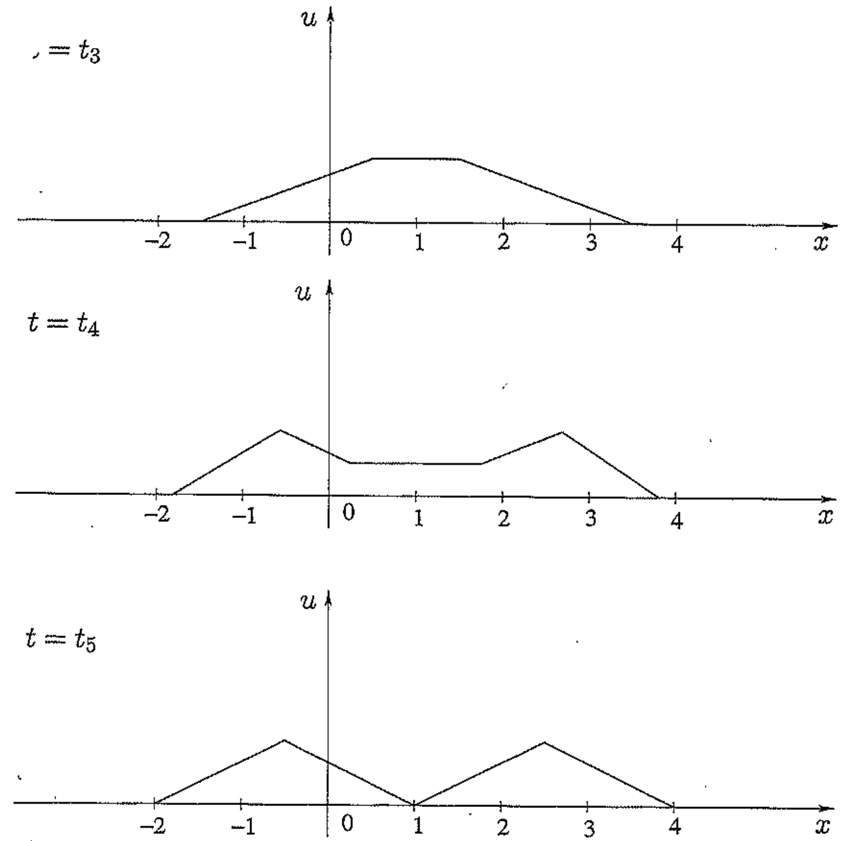


Figure 17.4

(c) Note that the right-moving signal will reach the point $\overline{x} = 11$ when $x + 2t = 11$, and this characteristic hits the t-axis when $t = \frac{11}{2}$.

Exercise 17.13. ✗✗✗✗

The following hyperbolic model arises as the equation for the moment generating function $A(s,t)$ of a stochastic birth-death process.

$$\frac{\partial A}{\partial t}(s,t) - (s-1)(bs - \delta(t))\frac{\partial A}{\partial s}(s,t) = 0, \tag{17.7}$$

$$A(s,0) = s^{n_0},$$

where b denotes a constant birth rate and $\delta(t)$ a time-dependent death rate. Find $A(s,t)$.

Solution. We use the method of characteristics to solve for $A(s,t)$. Note that if $s = s(t)$ is a curve in the st-plane such that

$$\frac{ds}{dt} = -(s-1)\,(bs - \delta(t)), \tag{17.8}$$

then from the chain rule, along this curve the partial differential equation (17.7) becomes

$$\frac{d}{dt}\left[A(s(t),t)\right] = \frac{\partial A}{\partial t}(s(t),t) + \frac{\partial A}{\partial s}(s(t),t)\cdot\frac{ds}{dt} = 0,$$

that is,

$$A(s(t),t) = \text{ constant } = A(s(0),0) = s(0)^{n_0},$$

and we need only solve the characteristic equation

$$\frac{ds(t)}{dt} = -(s-1)\,(bs-\delta(t)) \tag{17.9}$$

for the equation of the curve passing through the point $(s(0),0)$. The differential equation (17.9) can be written as

$$\begin{aligned}\frac{ds}{dt} &= (1-s)\,(bs-\delta(t))\\ &= (1-s)\,(-b+bs+b-\delta(t))\\ &= (1-s)\,(b-\delta(t)) - b(1-s)^2,\end{aligned}$$

that is,

$$\frac{1}{(1-s)^2}\,\frac{ds}{dt} = \frac{b-\delta(t)}{1-s} - b,$$

and letting $y(t) = 1/[1-s(t)]$, we have

$$\frac{dy}{dt} - (b-\delta(t))y = -b. \tag{17.10}$$

This is a first-order linear differential equation for $y = y(t)$ which can be solved using an integrating factor or the method of variation of parameters. Multiplying by the integrating factor

$$\Lambda(t) = \exp\left(-\int_0^t (b-\delta(\tau))\,d\tau\right),$$

we have

$$\frac{d}{dt}\left[\Lambda(t)y(t)\right] = -b\Lambda(t).$$

Integrating and using the fact that $\Lambda(0) = 1$, we have

$$\Lambda(t)y(t) = y(0) - b\int_0^t \Lambda(\tau)\,d\tau.$$

Therefore,

$$\frac{\Lambda(t)}{1-s(t)} = \frac{1}{1-s(0)} - b\int_0^t \Lambda(\tau)\,d\tau,$$

and we can solve this equation for $s(t)$ and obtain an explicit form for the equation of the characteristic curve $s = s(t)$. However, since

$$A(s(t),t) = \text{ constant } = A(s(0),0) = s(0)^{n_0}, \tag{17.11}$$

we need to find the anchor point $(s(0),0)$. We solve Equation 17.11 for $s(0)$ to get

$$s(0) = 1 - \frac{1}{\dfrac{\Lambda(t)}{1-s(t)} + b\displaystyle\int_0^t \Lambda(\tau)\,d\tau},$$

and the solution is constant along the characteristic $s = s(t)$, so that

$$A(s(t),t) = A(s(0),0) = s(0)^{n_0} = \left[1 - \frac{1}{\dfrac{\Lambda(t)}{1-s(t)} + b\displaystyle\int_0^t \Lambda(\tau)\,d\tau}\right]^{n_0} \tag{17.12}$$

where

$$\Lambda(t) = \exp\left(-\int_0^t (b-\delta(\tau))\,d\tau\right).$$

Since the initial value problem

$$\frac{dy}{dt} - (b-\delta(t))y = -b,$$

$$y(0) = \frac{1}{1-s(0)}$$

has a unique solution, given a point (s,t) in the st-plane, there exists a unique characteristic passing through this point with $s(0) = s_0$, and

$$A(s,t) = A(s_0,0) = {s_0}^{n_0} = \left[1 - \frac{1}{\dfrac{\Lambda(t)}{1-s} + b\displaystyle\int_0^t \Lambda(\tau)\,d\tau}\right]^{n_0}, \tag{17.13}$$

where

$$\Lambda(t) = \exp\left(-\int_0^t (b-\delta(\tau))\,d\tau\right).$$

This can be simplified somewhat by substituting the expression for $\Lambda(t)$, and we find

$$A(s,t) = \left[1 - \frac{\exp\left(\displaystyle\int_0^t (b-\delta(z))\,dz\right)}{\dfrac{1}{1-s} + b\displaystyle\int_0^t \exp\left[\int_\tau^t (b-\delta(z))\,dz\right]d\tau}\right]^{n_0}. \tag{17.14}$$

Exercise 17.14. ✗✗
Show that the first-order partial differential equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = u, \quad a, b \neq 0 \tag{17.15}$$

has no bounded solutions other than the trivial solution.

Solution. In Exercise 17.10 we found the general solution to the partial differential equation (17.15) using the method of characteristics. Here we present a slightly different argument, originally given as the solution to Problem A3 in the 2011 Putnam competition (see *The American Mathematical Monthly*, 118, **8**, October 2011, p. 730).

Suppose that $u = u(x,t)$ is a solution to (17.15) and there is a constant $M > 0$ such that

$$|u(x,t)| \leqslant M \tag{17.16}$$

for all $x, t \in \mathbb{R}$. Let x_0 and t_0 be fixed real numbers, and let

$$x = x_0 + bs \qquad \text{and} \qquad t = t_0 + as,$$

and define

$$h(s) = u(x_0 + bs, t_0 + as)$$

for $s \in \mathbb{R}$.
From the chain rule and the partial differential equation (17.15), we have

$$\begin{aligned} h'(s) &= \frac{\partial u}{\partial t}(x_0 + bs, t_0 + as) \cdot \frac{dt}{ds} + \frac{\partial u}{\partial x}(x_0 + bs, t_0 + as) \cdot \frac{dx}{ds} \\ &= a\frac{\partial u}{\partial t}(x_0 + bs, t_0 + as) + b\frac{\partial u}{\partial x}(x_0 + bs, t_0 + as) \\ &= u(x_0 + bs, t_0 + as) \\ &= h(s) \end{aligned}$$

for $s \in \mathbb{R}$. Thus, h satisfies the first-order differential equation

$$h'(s) = h(s),$$

with solution $h(s) = h(0)\, e^s$, and from (17.16), we have

$$|h(0)| = |h(s)|\, e^{-s} \leqslant Me^{-s}$$

for all $s \in \mathbb{R}$. Therefore,

$$h(0) = u(x_0, t_0) = 0,$$

and since $(x_0, t_0) \in \mathbb{R}^2$ was arbitrary, $u(x,t) \equiv 0$, and the partial differential equation (17.15) has no nontrivial bounded solutions.