Partial Differential Equations — Assignment 3.2

Spring 2017 — Due February 2

Exercise 1. If u is a function of x and t with continuous second order partial derivatives, and we set

$$\alpha = ax + bt, \quad \beta = mx + nt$$

with $an - bm \neq 0$, use the chain rule to show that

$$u_{tt} = b^2 u_{\alpha\alpha} + 2bn u_{\alpha\beta} + n^2 u_{\beta\beta}.$$

Exercise 2. Show that if $u(x,t) = F(x+ct) + G(x-ct)$ satisfies

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad \text{for all } x \in \mathbb{R},$$

then

$$\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} = \frac{1}{2c} \begin{pmatrix} c & 1 \\ c & -1 \end{pmatrix} \begin{pmatrix} f(x) \\ \int g(x)\, dx \end{pmatrix}.$$

Exercise 3. Continuing the notation of the preceding exercise, if $G_1(x)$ and $G_2(x)$ are both antiderivatives of $g(x)$, show that $u(x,t)$ does not depend on which one we choose to represent $\int g(x)\, dx$. [*Suggestion:* Write down an equation relating G_1 and G_2.]

Exercise 4. Use exercises 2 and 3 to help you solve the 1-D wave equation (on the domain $\mathbb{R} \times [0,\infty)$) subject to the given initial data.

a. $u(x,0) = f(x)$, $u_t(x,0) = 0$

b. $u(x,0) = \dfrac{1}{1+x^2}$, $u_t(x,0) = -2xe^{-x^2}$

c. $u(x,0) = e^{-x^2}$, $u_t(x,0) = \dfrac{x}{(1+x^2)^2}$

Exercise 5. Suppose that $u(x,t)$ and $v(x,t)$ have continuous second order partial derivatives and are related through the equations

$$\frac{\partial u}{\partial t} = -A\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} = -B\frac{\partial u}{\partial x}$$

for some positive constants A and B. Show that u and v are both solutions of the 1-D wave equation with $c = \sqrt{AB}$.