Solving First Order PDEs

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Partial Differential Equations
Lecture 2

Solving the transport equation

Goal: Determine every function $u(x, t)$ that solves

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0,$$

where v is a fixed constant.

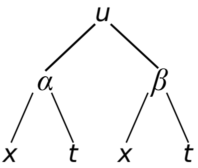
Idea: Perform a *linear change of variables* to eliminate one partial derivative:

$$\alpha = ax + bt,$$
$$\beta = cx + dt,$$

where:

$$x, t : \text{original independent variables,}$$
$$\alpha, \beta : \text{new independent variables,}$$
$$a, b, c, d : \text{constants to be chosen ``conveniently,''}$$
$$\text{must satisfy } ad - bc \neq 0.$$

We use the *multivariable chain rule* to convert to α and β derivatives:



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}.$$

Hence

$$\begin{aligned}\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} &= \left(b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}\right) + v\left(a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}\right)\\ &= (b+av)\frac{\partial u}{\partial \alpha} + (d+cv)\frac{\partial u}{\partial \beta}.\end{aligned}$$

Choosing $a = 0, b = 1, c = 1, d = -v$, the original PDE becomes

$$\frac{\partial u}{\partial \alpha} = 0.$$

This tells us that

$$u = f(\beta) = f(cx + dt) = f(x - vt)$$

for *any* (differentiable) function f.

Theorem

The general solution to the transport equation $\dfrac{\partial u}{\partial t} + v\dfrac{\partial u}{\partial x} = 0$ *is given by*

$$u(x, t) = f(x - vt),$$

where f is any *differentiable function of one variable.*

Example

Solve the transport equation $\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$ *given the* initial condition

$$u(x,0) = xe^{-x^2}, \ -\infty < x < \infty.$$

Solution: We know that the general solution is given by

$$u(x,t) = f(x-3t).$$

To find f we use the initial condition:

$$f(x) = f(x - 3\cdot 0) = u(x,0) = xe^{-x^2}.$$

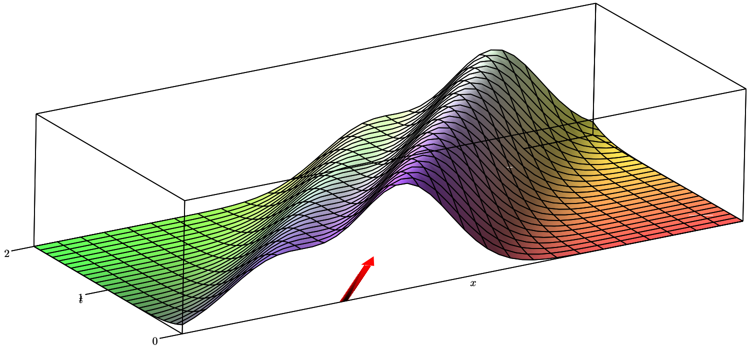
Thus

$$u(x,t) = (x-3t)e^{-(x-3t)^2}.$$

Interpreting the solutions of the transport equation

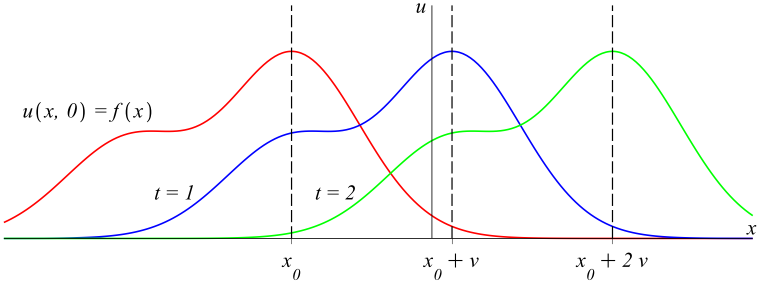
In three dimensions (*xtu*-space):

- The graph of the solution is the surface obtained by translating $u = f(x)$ along the vector $\mathbf{v} = \langle v, 1 \rangle$;
- The solution is constant along lines (in the *xt*-plane) parallel to $\mathbf{v}$.



If we plot the solution $u(x,t) = f(x - vt)$ in the xu-plane, and animate t:

- $f(x) = u(x,0)$ is the *initial condition* (concentration);
- $u(x,t)$ is a *traveling wave* with velocity v and shape given by $u = f(x)$.



In general: a linear change of variables can always be used to convert a PDE of the form

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = C(x, y, u)$$

into an "ODE," i.e. a PDE containing only one partial derivative.

Example

Solve $5\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x$ *given the initial condition*

$$u(x, 0) = \sin 2\pi x, \ -\infty < x < \infty.$$

Solution: As above, we perform the linear change of variables

$$\alpha = ax + bt,$$
$$\beta = cx + dt.$$

We find that

$$5\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=5\left(b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}\right)+\left(a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\right)$$
$$=(a+5b)\frac{\partial u}{\partial \alpha}+(c+5d)\frac{\partial u}{\partial \beta}.$$

We choose $a=1,\ b=0,\ c=5,\ d=-1.$ Note that

$$ad-bc=-1\neq 0,$$
$$\alpha=ax+bt=x,$$
$$\beta=cx+dt=5x-t.$$

So the PDE (in the variables α, β) becomes

$$\frac{\partial u}{\partial \alpha}=\alpha.$$

Integrating with respect to α yields

$$u = \frac{\alpha^2}{2} + f(\beta) = \frac{x^2}{2} + f(5x - t).$$

The initial condition tells us that

$$\frac{x^2}{2} + f(5x) = u(x, 0) = \sin 2\pi x.$$

If we replace x with $x/5$, we get

$$f(x) = \sin \frac{2\pi x}{5} - \frac{x^2}{50}.$$

Therefore

$$\begin{aligned} u(x,t) &= \frac{x^2}{2} + f(5x-t) \\ &= \frac{x^2}{2} + \sin\frac{2\pi(5x-t)}{5} - \frac{(5x-t)^2}{50} \\ &= \frac{xt}{5} - \frac{t^2}{50} + \sin\frac{2\pi(5x-t)}{5}. \end{aligned}$$

Remark: There are an infinite number of choices for a, b, c, d that will "correctly" eliminate either α or β from the PDE. Although they may appear different, the solutions obtained are always independent of the choice made.

Characteristic curves

Goal: Develop a technique to solve the (somewhat more general) first order PDE

$$\frac{\partial u}{\partial x} + p(x, y)\frac{\partial u}{\partial y} = 0. \tag{1}$$

Idea: Look for *characteristic curves* in the xy-plane along which the solution u satisfies an ODE.

Consider u along a curve $y = y(x)$. On this curve we have

$$\frac{d}{dx}u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \tag{2}$$

Comparing (1) and (2), if we require

$$\frac{dy}{dx} = p(x, y), \tag{3}$$

then the PDE becomes the ODE

$$\frac{d}{dx} u(x, y(x)) = 0. \tag{4}$$

These are the *characteristic ODEs* of the original PDE.

If we express the general solution to (3) in the form $\varphi(x, y) = C$, each value of C gives a characteristic curve.

Equation (4) says that u is constant along the characteristic curves, so that

$$u(x, y) = f(C) = f(\varphi(x, y)).$$

The Method of Characteristics - Special Case

Summarizing the above we have:

Theorem

The general solution to

$$\frac{\partial u}{\partial x} + p(x,y)\frac{\partial u}{\partial y} = 0$$

is given by

$$u(x,y) = f(\varphi(x,y)),$$

where:

- $\varphi(x,y) = C$ *gives the general solution to* $\frac{dy}{dx} = p(x,y)$*, and*
- f *is any differentiable function of one variable.*

Example

Solve $2y\frac{\partial u}{\partial x} + (3x^2 - 1)\frac{\partial u}{\partial y} = 0$ *by the method of characteristics.*

Solution: We first divide the PDE by $2y$ obtaining

$$\frac{\partial u}{\partial x} + \underbrace{\frac{3x^2 - 1}{2y}}_{p(x,y)} \frac{\partial u}{\partial y} = 0.$$

So we need to solve

$$\frac{dy}{dx} = \frac{3x^2 - 1}{2y}.$$

This is separable:

$$2y\,dy = 3x^2 - 1\,dx.$$

$$\int 2y\,dy = \int 3x^2 - 1\,dx$$
$$y^2 = x^3 - x + C.$$

We can put this in the form $y^2 - x^3 + x = C$ and hence

$$u(x,y) = f\left(y^2 - x^3 + x\right).$$

Remark: This technique can be generalized to PDEs of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u).$$

Example

Solve $\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = u.$

As above, along a curve $y = y(x)$ we have

$$\frac{d}{dx}u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

Comparison with the original PDE gives the characteristic ODEs

$$\frac{dy}{dx} = x,$$
$$\frac{d}{dx}u(x, y(x)) = u(x, y(x)).$$

The first tells us that

$$y = \frac{x^2}{2} + y(0),$$

and the second that [1]

$$u(x, y(x)) = u(0, y(0))e^x = f(y(0))e^x.$$

Combining these gives

$$u(x, y) = f\left(y - \frac{x^2}{2}\right) e^x.$$

[1] Recall that the solution to the ODE $\frac{dw}{dx} = kw$ is $w = Ce^{kx}$. Since $w(0) = Ce^0 = C$, we can write this as $w = w(0)e^{kx}$.

Summary

Consider a first order PDE of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u). \tag{5}$$

- When $A(x,y)$ and $B(x,y)$ are *constants*, a linear change of variables can be used to convert (5) into an "ODE."
- In general, the method of characteristics yields a system of ODEs equivalent to (5).

In principle, these ODEs can always be solved completely to give the general solution to (5).