Introduction to the Wave Equation

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Partial Differential Equations
Lecture 4

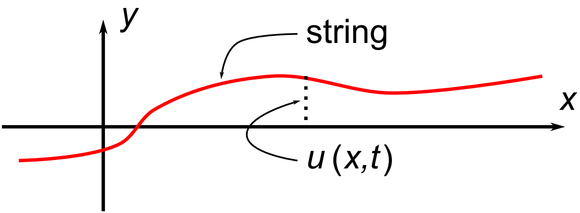
Modeling the Motion of an Ideal Elastic String

Idealizing Assumptions:

- The only force acting on the string is (constant) tension, i.e. no friction, resistance to bending, etc.
- The string's motion takes place in a single plane.
- The displacement of the string from equilibrium is small relative to its length, i.e. only small deflections and no stretching.

Set-up: Place the string in the xy-plane, along the x-axis at rest, and let

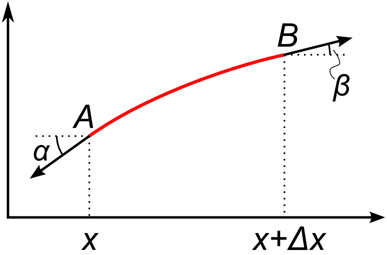
$$u(x,t) = \begin{cases} \text{displacement of string from rest} \\ \text{at position } x \text{ and time } t. \end{cases}$$



We furthermore set

$$\tau = \begin{cases} \text{(constant) magnitude of tension} \\ \text{throughout string (units = force),} \end{cases}$$

$$\rho = \begin{cases} \text{(constant) linear mass density} \\ \text{throughout string (units = mass/length).} \end{cases}$$



Tensions on small string segment

By Newton's Second Law

$$\underbrace{-\tau\sin\alpha+\tau\sin\beta}_{\text{vertical components of tension}} \approx \overbrace{\rho\,\Delta x}^{\text{mass of segment}} \underbrace{u_{tt}(x,t)}_{\text{vertical acceleration}}.$$

Since α is small

$$\sin\alpha \approx \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{slope at } A = u_x(x,t).$$

Likewise $\sin\beta \approx \text{slope at } B = u_x(x+\Delta x, t)$.

Plugging this into the Second Law expression gives

$$\tau\left(\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}\right)\approx\rho\,u_{tt}(x,t).$$

Letting $\Delta x\to 0$ we obtain the *exact* expression

$$\tau\,u_{xx}=\rho\,u_{tt}.$$

Finally, setting $c^2=\tau/\rho$, we obtain the *one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}.$$

Remarks.

- The units of c^2 are

$$\frac{\text{force}}{\text{mass}/\text{length}} = \frac{\text{mass} \cdot \text{length}/\text{time}^2}{\text{mass}/\text{length}} = \left(\frac{\text{length}}{\text{time}}\right)^2,$$

  so that c has the units of speed.

- Modified assumptions yield modified PDEs, e.g.
  - $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + \dfrac{F(x,t)}{\rho}$ [$F(x,t) =$ external force/length]
  - $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - 2k\dfrac{\partial u}{\partial t}$ [Fluid resistance $\propto$ velocity]

Solving the (unrestricted) 1-D wave equation

If we impose no additional restrictions, we can derive the general solution to the 1-D wave equation.

We perform the linear change of variables

$$\alpha = ax + bt,$$
$$\beta = mx + nt,$$
$$(an - bm \neq 0).$$

The chain rule (applied twice) gives

$$u_{tt} = b^2 u_{\alpha\alpha} + 2bn u_{\alpha\beta} + n^2 u_{\beta\beta},$$
$$u_{xx} = a^2 u_{\alpha\alpha} + 2am u_{\alpha\beta} + m^2 u_{\beta\beta}.$$

The wave equation $u_{tt} - c^2 u_{xx} = 0$ then becomes

$$(b^2 - a^2c^2)u_{\alpha\alpha} + 2(bn - amc^2)u_{\alpha\beta} + (n^2 - c^2m^2)u_{\beta\beta} = 0.$$

Choosing $a = m = 1$, $b = c$ and $n = -c$ results in

$$-4c^2 u_{\alpha\beta} = 0 \iff \frac{\partial^2 u}{\partial\alpha\partial\beta} = 0.$$

Now integrate with respect to β, then α:

$$\frac{\partial u}{\partial \alpha} = \int \frac{\partial^2 u}{\partial\alpha\partial\beta}\,d\beta = \int 0\,d\beta = \overbrace{f(\alpha)}^{\text{"constant" of integration}},$$

$$u = \int \frac{\partial u}{\partial \alpha}\,d\alpha = \int f(\alpha)\,d\alpha = \underbrace{F(\alpha)}_{\text{antideriv. of } f} + \underbrace{G(\beta)}_{\text{"constant" of integration}}.$$

Back substitution then gives:

Theorem

The general solution to the 1-D wave equation is

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary (twice-differentiable) functions of one variable.

Remarks:

- The solution consists of the superposition of two traveling waves with speed c, but moving in opposite directions.
- The functions F and G (and hence the solution u) are completely determined by initial data of the form

$$u(x,0) = f(x) \ \text{[the \textit{initial shape} of the string]},$$
$$u_t(x,0) = g(x) \ \text{[the \textit{initial velocity} of the string]}.$$

Example

Find the solution of the 1-D wave equation that satisfies

$$u(x,0) = e^{-x^2}, \;\; u_t(x,0) = 0.$$

We must solve for F, G in the general solution

$$u(x,t) = F(x+ct) + G(x-ct).$$

Since

$$u_t(x,t) = cF'(x+ct) - cG'(x-ct),$$

from the initial conditions we obtain

$$F(x) + G(x) = u(x,0) = e^{-x^2},$$
$$cF'(x) - cG'(x) = u_t(x,0) = 0.$$

The second equation implies that

$$F'(x) = G'(x) \implies F(x) = G(x) + K.$$

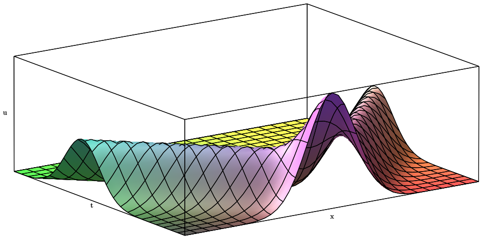
Substituting this into the first equation we find

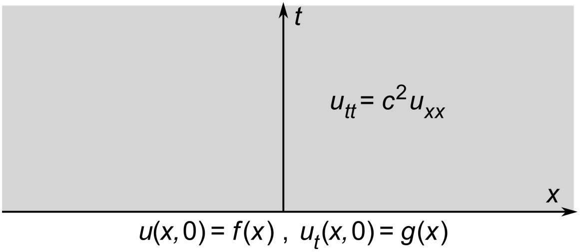
$$\begin{aligned} 2G(x) + K = e^{-x^2} &\implies G(x) = \frac{1}{2}e^{-x^2} - \frac{K}{2} \\ &\implies F(x) = \frac{1}{2}e^{-x^2} + \frac{K}{2}. \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u(x,t) &= F(x+ct) + G(x-ct) \\ &= \left(\frac{1}{2}e^{-(x+ct)^2} + \frac{K}{2}\right) + \left(\frac{1}{2}e^{-(x-ct)^2} - \frac{K}{2}\right) \\ &= \frac{1}{2}\left(e^{-(x+ct)^2} + e^{-(x-ct)^2}\right). \end{aligned}$$

The solution surface and its domain





Remarks: The domain of $u(x,t)$ is

$$H = \mathbb{R} \times [0, \infty).$$

1. Spatiotemporally this means:
   * The string is *infinitely long*.
   * We only let time move *forward* from $t = 0$.
2. The function $u(x,t)$ satisfies:

$$\left.\begin{array}{l} * \ u_{tt} = c^2 u_{xx} \text{ on the } \textit{interior} \text{ of } H; \\ * \ u = f,\ u_t = g \text{ on the } \textit{boundary} \text{ of } H. \end{array}\right\} \textit{Boundary Value Problem} \text{ (BVP) on } H.$$

Two more physical constraints

We now assume that:

- The string has finite length L.
- The string is fixed at both ends.

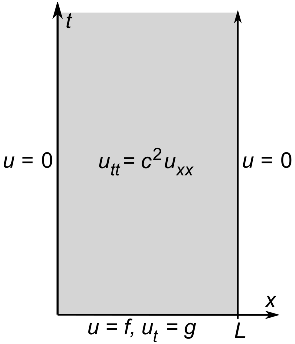
These constraints yield the modified BVP

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x),$$

$$u_t(x,0) = g(x),$$



with domain $S = [0,L] \times [0,\infty)$.

Solving the fixed endpoint string problem

Since we do not have initial data along the *entire* x-axis, we *cannot* simply appeal to the previous solution.[1]

It's relatively easy to find solutions to the 1-D wave equation that satisfy $u(0,t) = u(L,t) = 0$, e.g.

$$u(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right),$$

$$u(x,t) = \frac{L}{c\pi}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{c\pi t}{L}\right).$$

But these come with their own prescribed initial behavior, and we want to be able to choose $u(x,0)$ and $u_t(x,0)$ *arbitrarily.*

[1] We'll see later that there's actually a clever way around this issue.

What now?

We will use the following fact to build solutions with desired initial behavior from simple solutions like those above.

Theorem (The Principle of Superposition)

If u_1, u_2 are solutions of the 1-D wave equation, then so is $u = c_1 u_1 + c_2 u_2$ for any *choice of constants c_1 and c_2. If, in addition, $u_1 = u_2 = 0$ on the vertical edges of S, then $u = 0$ on the vertical edges as well.*

That is, if $u_1(x, t)$ and $u_2(x, t)$ describe the motion of a vibrating string of length L with fixed end points, then so does any linear combination of them.

Remarks:

- Note that initial behavior is *not* addressed.
- This generalizes to any homogeneous linear PDE.

Proof of the principle of superposition

If u_1, u_2 both solve the 1-D wave equation and c_1, c_2 are constants, then

$$\begin{aligned}(c_1 u_1 + c_2 u_2)_{tt} &= c_1(u_1)_{tt} + c_2(u_2)_{tt}\\ &= c_1 c^2 (u_1)_{xx} + c_2 c^2 (u_2)_{xx}\\ &= c^2(c_1 u_1 + c_2 u_2)_{xx},\end{aligned}$$

so that $u = c_1 u_1 + c_2 u_2$ is also a solution.

Furthermore, if $u_1 = u_2 = 0$ on the vertical edges of S, then certainly $u = c_1 u_1 + c_2 u_2 = 0$ there as well. QED.

Remark: The Principle of Superposition is easily seen to hold for linear combinations of *any number* of solutions.

Example

We saw above that

$$u_1(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right),$$
$$u_2(x,t) = \frac{L}{c\pi}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{c\pi t}{L}\right)$$

both solve the fixed endpoint length L vibrating string problem.

By the Principle of Superposition, so do

$$u = 2u_1 - u_2 = \sin\left(\frac{\pi x}{L}\right)\left(2\cos\left(\frac{c\pi t}{L}\right) - \frac{L}{c\pi}\sin\left(\frac{c\pi t}{L}\right)\right),$$
$$u = -\sqrt{2}u_2 + \pi u_2 = \sin\left(\frac{\pi x}{L}\right)\left(-\sqrt{2}\cos\left(\frac{c\pi t}{L}\right) + \frac{L}{c}\sin\left(\frac{c\pi t}{L}\right)\right).$$

Example

More generally, for $n = 1, 2, 3, \ldots$ the functions

$$u_n(x,t) = \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$
$$v_n(x,t) = \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$

solve the fixed endpoint length L vibrating string problem.

By the Principle of Superposition, so does the function

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty}\left(a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

These series solutions satisfy the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad [= f(x)?],$$
$$u_t(x,0) = \sum_{n=1}^{\infty} a_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \quad [= g(x)?].$$

So, one can solve the vibrating string problem with initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x), 0 \le x \le L,$$

provided that $f(x)$ and $g(x)$ can be expressed as (possibly infinite) linear combinations of the functions $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$

These are examples of *Fourier series*.

Questions to be addressed

In terms of solving the finite vibrating string problem, we are now faced with:

- Which functions are expressible as Fourier series?
- How can we find the Fourier series expansion of a given function?

Once we've pinned these down, we'll return to ask:

- Where did the "simple" solutions come from, and are there others?
- How are "simple" solutions found for other linear PDEs, and how do Fourier series generalize to these?

This is what we'll spend most of the rest of the semester thinking about!