More on Fourier Series

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Partial Differential Equations
Lecture 6.1

New Fourier series from old

Recall: Given a function $f(x)$, we can dilate/translate its graph via multiplication/addition, as follows.

Geometric operation	Mathematical implementation
Dilate along the x-axis by a factor of a	$f(x/a)$
Dilate along the y-axis by a factor of b	$bf(x)$
Translate (right) along the x-axis by c units	$f(x-c)$
Translate (up) along the y-axis by d units	$f(x)+d$

One has the following general principles.

Theorem

If the graph of $f(x)$ is obtained from $g(x)$ by dilations and/or translations, then the same operations can be used to obtain the Fourier series of f from that of g.

Theorem

If $f(x)$ is a linear combination of $g_1(x), g_2(x), \ldots, g_n(x)$, then the Fourier series of f is the same linear combination of the Fourier series of $g_1, g_2, \ldots, g_n$.

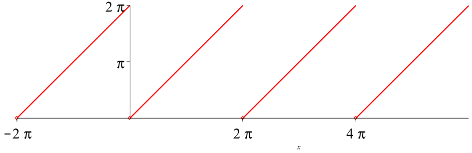
Remarks:

- These are both easily derived from Euler's formulas for the Fourier coefficients.
- These tell us that we can construct Fourier series of "new" functions from existing series.

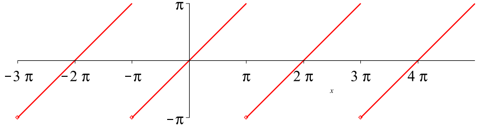
Example

Use an existing series to find the Fourier series of the 2π-periodic function given by $f(x) = x$ for $0 \leq x < 2\pi$.

The graph of $f(x)$:



This function can be obtained from the earlier sawtooth wave



by translating both up and to the right by π units.

The old sawtooth wave has Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(nx)}{n},$$

so the function f has Fourier series

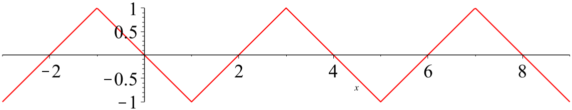
$$\begin{aligned}
&\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n(x-\pi))}{n}\\
&=\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\sin(nx)\cos(n\pi)-\sin(n\pi)\cos(nx)\right)\\
&=\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^{n}}{n}\sin(nx)\\
&=\pi-2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}
\end{aligned}$$

Example

Use an existing series to find the Fourier series of the 4-periodic function satisfying

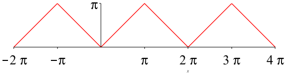
$$f(x) = \begin{cases} -x & \text{if } -1 \leq x < 1 \\ x-2 & \text{if } 1 \leq x < 3 \end{cases}.$$

The graph of $f(x)$:

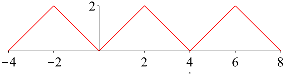


We can obtain f from the graph of an earlier 2π-periodic triangular wave.

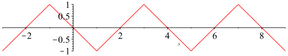
Earlier wave: $g(x)$



Dilation of $2/\pi$ along both axes: $\frac{2}{\pi} g\left(\frac{\pi x}{2}\right)$



Translation by 1 along both axes: $-1 + \frac{2}{\pi} g\left(\frac{\pi(x-1)}{2}\right)$



We already know that the Fourier series for g is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

We simply transform it as above, and simplify.

This yields

$$-1+\frac{2}{\pi}\left(\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)\pi(x-1)/2)}{(2k+1)^2}\right)$$

The cosine term inside the sum is

$$\begin{aligned}\cos\left(\frac{(2k+1)\pi x}{2}-\frac{(2k+1)\pi}{2}\right) &= \cos\left(\frac{(2k+1)\pi x}{2}\right)\cos\left(\frac{(2k+1)\pi}{2}\right) \\ &\quad + \sin\left(\frac{(2k+1)\pi x}{2}\right)\sin\left(\frac{(2k+1)\pi}{2}\right) \\ &= (-1)^k \sin\left(\frac{(2k+1)\pi x}{2}\right).\end{aligned}$$

So the series simplifies to

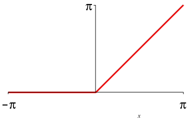
$$-\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Example

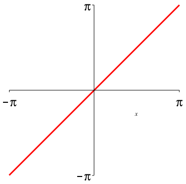
Use existing series to find the Fourier series of the 2π-periodic function satisfying

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0, \\ x & \text{if } 0 \le x < \pi. \end{cases}$$

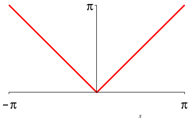
The graph of $f(x)$ (left) is the average of the sawtooth and triangular waves shown.



$= \frac{1}{2}$



$+ \frac{1}{2}$



So, the Fourier series of f is the average of our two previous series:

$$\frac{1}{2}\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)+\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)x)}{(2k+1)^2}\right)$$

$$=\frac{\pi}{4}-\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)x)}{(2k+1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx).$$

We could combine these into one series, but it's easier to just leave the cosine and sine series separate.

Differentiating Fourier series

Term-by-term differentiation of a series can be a useful operation, *when it is valid*. The following result tells us when this is the case with Fourier series.

Theorem

Suppose f is 2π-periodic and piecewise smooth. If f' is also piecewise smooth, and f is continuous everywhere, then the Fourier series for f' can be obtained from that of f using term-by-term differentiation.

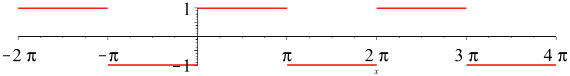
Remark: This can be proven by using integration by parts in the Euler formulas for the Fourier coefficients of f'.

Example

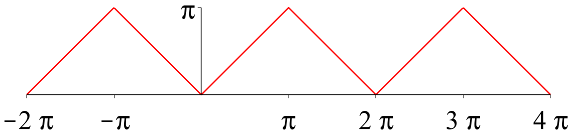
Use an existing series to find the Fourier series of the 2π-periodic function satisfying

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x < 0, \\ 1 & \text{if } 0 \le x < \pi. \end{cases}$$

The graph of $f(x)$ (a *square wave*)



shows that it is the derivative of the triangular wave.



Since the triangular wave is continuous everywhere, we can differentiate its Fourier series term-by-term to get the series for the square wave.

$$\frac{d}{dx}\left(\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)x)}{(2k+1)^2}\right)=-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{-(2k+1)\sin((2k+1)x)}{(2k+1)^2}$$
$$=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{(2k+1)}.$$

Warning: The hypothesis that f is continuous is *extremely important*. For example, if we term-wise differentiate the Fourier series for the *discontinuous* square wave (above), we get

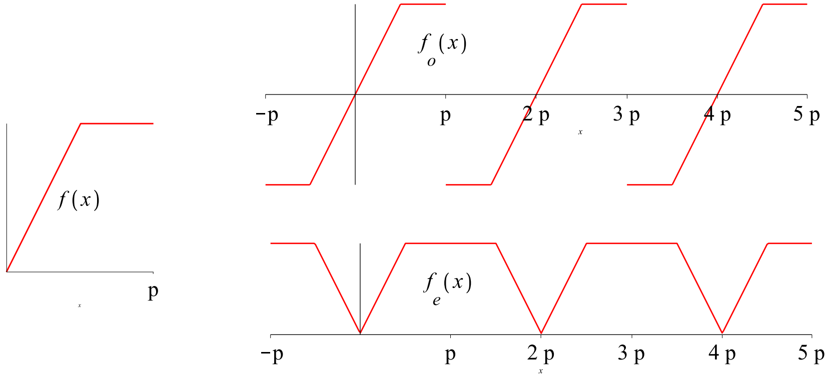
$$\frac{4}{\pi}\sum_{k=0}^{\infty}\cos((2k+1)x)$$

which converges (almost) *nowhere*!

Half-range expansions

Goal: Given a function $f(x)$ defined for $0 \leq x \leq p$, write $f(x)$ as a linear combination of sines and cosines.

Idea: Extend f to have period $2p$, and find the Fourier series of the resulting function.



Sine and cosine series

We set

$$f_o = \text{odd } 2p\text{-periodic extension of } f,$$
$$f_e = \text{even } 2p\text{-periodic extension of } f.$$

If we expand f_o as a Fourier series, it will involve only sines:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right).$$

This is the *sine series expansion* of f.

According to Euler's formula the Fourier coefficients are given by

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f_o(x)\sin\left(\frac{n\pi x}{p}\right)}_{\text{even}} dx = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right) dx.$$

If we expand f_e as a Fourier series, it will involve only cosines:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right).$$

This is the *cosine series expansion* of f.

This time Euler's formulas give

$$a_0 = \frac{1}{2p}\int_{-p}^{p} \underbrace{f_e(x)}_{\text{even}}\,dx = \frac{1}{p}\int_0^p f(x)\,dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f_e(x)\cos\left(\frac{n\pi x}{p}\right)}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

If f is piecewise smooth, both the sine and cosine series converge to the function $\dfrac{f(x+)+f(x-)}{2}$ (on the interval $[0,p]$).

Example

Find the sine and cosine series expansions of $f(x) = 3 - x$ on the interval $0 \leq x \leq 3$.

Taking $p = 3$ in our work above, the coefficients of the sine series are given by

$$b_n = \frac{2}{3}\int_0^3 (3-x)\sin\left(\frac{n\pi x}{3}\right)\,dx$$
$$= \frac{2}{3}\left(\frac{-3(3-x)}{n\pi}\cos\left(\frac{n\pi x}{3}\right) - \frac{9}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right)\Bigg|_0^3\right)$$
$$= \frac{2}{3}\cdot\frac{9}{n\pi}\cos(0) = \frac{6}{n\pi}.$$

So, the sine series is

$$\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{3}\right).$$

The cosine series coefficients are

$$a_0 = \frac{1}{3}\int_0^3 3 - x\,dx = \frac{1}{3}\left(3x - \frac{x^2}{2}\Big|_0^3\right) = \frac{3}{2}$$

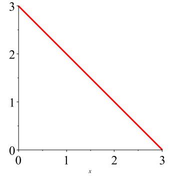
and for $n \geq 1$

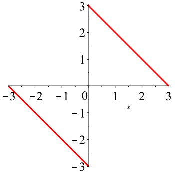
$$\begin{aligned} a_n &= \frac{2}{3}\int_0^3 (3-x)\cos\left(\frac{n\pi x}{3}\right)\,dx \\ &= \frac{2}{3}\left(\frac{3(3-x)}{n\pi}\sin\left(\frac{n\pi x}{3}\right) - \frac{9}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)\Big|_0^3\right) \\ &= \frac{2}{3}\left(-\frac{9}{n^2\pi^2}\cos(n\pi) + \frac{9}{n^2\pi^2}\right) = \begin{cases} \dfrac{12}{n^2\pi^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

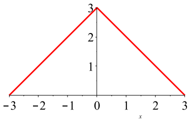
Since we can omit the terms with even n, we write $n = 2k + 1$ $(k \geq 0)$ and obtain the cosine series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) = \frac{3}{2} + \frac{12}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left(\frac{(2k+1)\pi x}{3}\right).$$

Here are the graphs of f, f_o and f_e (over one period):







Consequently, the sine series equals $f(x)$ for $0 < x \leq 3$, and the cosine series equals $f(x)$ for $0 \leq x \leq 3$.