Integral formulas for Fourier coefficients

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Partial Differential Equations
Lecture 6

Recall: Relative to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

the functions

$$1, \cos(x), \cos(2x), \cos(3x), \ldots \sin(x), \sin(2x), \sin(3x), \ldots$$

satisfy the *orthogonality relations*

$$\langle \cos(mx), \sin(nx)\rangle = 0,$$

$$\langle \cos(mx), \cos(nx)\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n \neq 0, \\ 2\pi & \text{if } m = n = 0, \end{cases}$$

$$\langle \sin(mx), \sin(nx)\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

By the linearity of the inner product, if

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

then

$$\begin{aligned}
\langle f(x), \cos(mx) \rangle &= a_0 \overbrace{\langle 1, \cos(mx) \rangle}^{=0 \text{ unless } m=0} + \sum_{n=1}^{\infty} \Big(a_n \overbrace{\langle \cos(nx), \cos(mx) \rangle}^{=0 \text{ unless } m=n} \\
&\qquad + b_n \underbrace{\langle \sin(nx), \cos(mx) \rangle}_{=0} \Big) \\
&= a_m \langle \cos(mx), \cos(mx) \rangle \\
&= \begin{cases} 2\pi a_0 & \text{if } m = 0, \\ \pi a_m & \text{if } m \neq 0. \end{cases}
\end{aligned}$$

Likewise, one can show that

$$\langle f(x), \sin(mx)\rangle = b_m \langle \sin(mx), \sin(mx)\rangle = \pi b_m.$$

Solving for the coefficients gives:

Theorem (Euler's Formulas)

If f is 2π-periodic and piecewise smooth, then its Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,\\
a_n &= \frac{\langle f(x), \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad (n \neq 0),\\
b_n &= \frac{\langle f(x), \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.
\end{aligned}$$

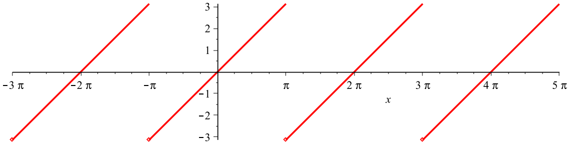
Remarks

- Technically we should have used $\frac{f(x+)+f(x-)}{2}$. However, the integrals cannot distinguish between this and $f(x)$.
- Because all the functions in question are 2π-periodic, we can integrate over *any* convenient interval of length 2π.
- If $f(x)$ is an odd function, so is $f(x)\cos(nx)$, and so $a_n = 0$ for all $n \geq 0$.
- If $f(x)$ is an even function, then $f(x)\sin(nx)$ is odd, and so $b_n = 0$ for all $n \geq 1$.

Example

Find the Fourier series for the 2π-periodic function that satisfies $f(x) = x$ for $-\pi < x \leq \pi$.

The graph of f (a *sawtooth wave*):



Because f is odd, we know

$$a_n = 0 \quad (n \geq 0).$$

According to Euler's formula:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx \\ &= \frac{1}{\pi}\left(\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\bigg|_{-\pi}^{\pi}\right) \\ &= \frac{1}{\pi}\left(\frac{-\pi\cos(n\pi)}{n} - \frac{\pi\cos(-n\pi)}{n}\right) \\ &= \frac{-2\cos(n\pi)}{n} = \frac{(-1)^{n+1}2}{n}. \end{aligned}$$

Therefore, the Fourier series of f is

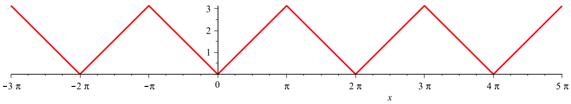
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n}\sin(nx) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(nx)}{n}.$$

Remark: Except where it is discontinuous, this series equals $f(x)$.

Example

Find the Fourier series of the 2π-periodic function satisfying $f(x) = |x|$ for $-\pi \leq x < \pi$.

The graph of f (a *triangular wave*):



This time, since f is even,

$$b_n = 0 \quad (n \geq 1).$$

By Euler's formula we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right|_0^{\pi} = \frac{\pi}{2}$$

and for $n \geq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|\cos(nx)}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \left.\frac{\cos(nx)}{n^2}\right|_0^{\pi}\right) = \frac{2}{\pi}\left(\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{2}{\pi n^2}((-1)^n - 1) = \begin{cases} \dfrac{-4}{\pi n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

In the Fourier series we may therefore omit the terms in which n is even, and assume that $n = 2k+1$, $k \geq 0$:

$$
\begin{aligned}
a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) &= \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2} \cos((2k+1)x) \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.
\end{aligned}
$$

Remarks:

- Since k is simply an index of summation, we are free to replace it with n again, yielding

$$
\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.
$$

- Because $f(x)$ is continuous everywhere, this equals $f(x)$ at all points.

Example

Use the result of the previous exercise to show that

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

If we set $x = 0$ in the previous example, we get

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(0)}{(2n+1)^2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

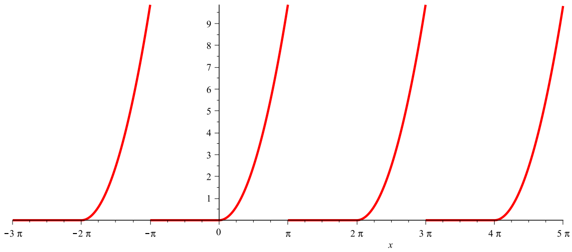
Solving for the series gives the result.

Remark: In Calculus II you learned that this series converges, but were unable to obtain its exact value.

Example

Find the Fourier series of the 2π-periodic function satisfying $f(x) = 0$ for $-\pi \leq x < 0$ and $f(x) = x^2$ for $0 \leq x < \pi$.

The graph of f:



Because f is neither even nor odd, we must compute all of its Fourier coefficients directly.

Since $f(x) = 0$ for $-\pi \le x < 0$, Euler's formulas become

$$a_0 = \frac{1}{2\pi}\int_0^\pi x^2\,dx = \frac{1}{2\pi}\left(\frac{x^3}{3}\bigg|_0^\pi\right) = \frac{\pi^2}{6}$$

and for $n \ge 1$

$$a_n = \frac{1}{\pi}\int_0^\pi x^2\cos(nx)\,dx = \frac{1}{\pi}\left(\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\bigg|_0^\pi\right)$$
$$= \frac{1}{\pi}\cdot\frac{2\pi\cos(n\pi)}{n^2} = \frac{2(-1)^n}{n^2},$$

$$b_n = \frac{1}{\pi}\int_0^\pi x^2\sin(nx)\,dx = \frac{1}{\pi}\left(-\frac{x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}\bigg|_0^\pi\right)$$
$$= \frac{1}{\pi}\left(-\frac{\pi^2\cos(n\pi)}{n} + \frac{2\cos(n\pi)}{n^3} - \frac{2}{n^3}\right) = \frac{(-1)^{n+1}\pi}{n} + \frac{2((-1)^n - 1)}{\pi n^3}.$$

Therefore the Fourier series of f is

$$\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}\cos(nx)+\left(\frac{(-1)^{n+1}\pi}{n}+\frac{2((-1)^n-1)}{\pi n^3}\right)\sin(nx)\right).$$

This will agree with $f(x)$ everywhere it's continuous.

Convergence of Fourier series

Given a Fourier series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{1}$$

let its Nth *partial sum* be

$$s_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) . \tag{2}$$

According to the definition of an infinite series, the Fourier series (1) is equal to

$$\lim_{N\to\infty} s_N(x).$$

According to the definition of the limit:

- We can *approximate* the (infinite) Fourier series (1) by the (finite) partial sums (2).

- The approximation of (1) by (2) improves (indefinitely) as we increase N.

Because $s_N(x)$ is a *finite* sum, we can use a computer to graph it.

In this way, we can visualize the convergence of a Fourier series.

Let's look at some examples...

These examples illustrate the following results. In both, $f(x)$ is 2π-periodic and piecewise smooth.

Theorem (Uniform convergence of Fourier series)

If $f(x)$ is continuous everywhere, then the partial sums $s_N(x)$ of its Fourier series converge uniformly *to $f(x)$ as $N \to \infty$.*
That is, by choosing N large enough we can make $s_N(x)$ arbitrarily close to $f(x)$ for all *x* simultaneously*.*

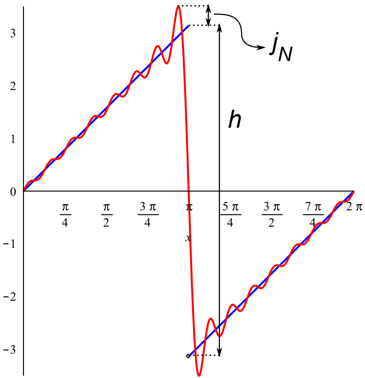
Theorem (Wilbraham-Gibbs phenomenon)

If $f(x)$ has a jump discontinuity at $x = c$, then the partial sums $s_N(x)$ of its Fourier series always "overshoot" $f(x)$ near $x = c$.
More precisely, as $N \to \infty$, the the ratio between the peak of the overshoot and the height of the jump tends to

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt - \frac{1}{2} = 0.08948\ldots \quad \text{(about 9\% of the jump).}$$

The Wilbraham-Gibbs phenomenon

A function $f(x)$ (in blue) with a jump discontinuity and a partial sum $s_N(x)$ (in red) of its Fourier series:



$$\lim_{N\to\infty} \frac{j_N}{h} = 0.08948\ldots$$

General Fourier series

If $f(x)$ is $2p$-periodic and piecewise smooth, then $\hat{f}(x) = f(px/\pi)$ has period $\frac{2p}{p/\pi} = 2\pi$, and is also piecewise smooth.

It follows that $\hat{f}(x)$ has a Fourier series:

$$\frac{\hat{f}(x+) + \hat{f}(x-)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Since $f(x) = \hat{f}(\pi x/p)$, we find that f also has a Fourier series:

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right).$$

We can use Euler's formulas to find a_n and b_n. For example

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right)\,dx = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt,$$

where in the final equality we used the substitution $t = px/\pi$.

In the same way one can show that for $n \geq 1$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\left(\frac{n\pi t}{p}\right)\,dt,$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\left(\frac{n\pi t}{p}\right)\,dt.$$

Since t is simply a "dummy" variable of integration, we may replace it with x in each case.

Remarks on general Fourier series

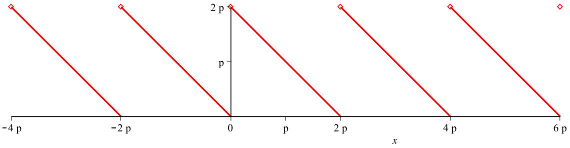
Everything we've done with 2π-periodic Fourier series continues to hold in this case, with p replacing π:

- We can compute general Fourier coefficients by integrating over any "convenient" interval of length $2p$.
- If p is left unspecified, then the formulae for a_n and b_n may involve p.
- If $f(x)$ is even, then $b_n = 0$ for all n.
- If $f(x)$ is odd, then $a_n = 0$ for all n.
- We still have the uniform convergence theorem and Wilbraham-Gibbs phenomenon.

Example

Find the Fourier series of the 2p-periodic function that satisfies $f(x) = 2p - x$ *for* $0 \le x < 2p$.

The graph of $f(x)$:



We will use Euler's formulas over the interval $[0, 2p]$ to simplify our calculations.

We have

$$a_0 = \frac{1}{2p}\int_0^{2p} 2p - x\,dx = \frac{1}{2p}\left(2px - \frac{x^2}{2}\bigg|_0^{2p}\right) = p$$

and for $n \geq 1$

$$\begin{aligned} a_n &= \frac{1}{p}\int_0^{2p}(2p-x)\cos\left(\frac{n\pi x}{p}\right)\,dx \\ &= \frac{1}{p}\left(\frac{p(2p-x)\sin(\frac{n\pi x}{p})}{n\pi} - \frac{p^2\cos(\frac{n\pi x}{p})}{n^2\pi^2}\Bigg|_0^{2p}\right) \\ &= \frac{1}{p}\left(-\frac{p^2\cos(2n\pi)}{n^2\pi^2} + \frac{p^2}{n^2\pi^2}\right) = 0, \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{p}\int_0^{2p}(2p-x)\sin\left(\frac{n\pi x}{p}\right)\,dx \\
&= \frac{1}{p}\left(\frac{-p(2p-x)\cos(\frac{n\pi x}{p})}{n\pi} - \frac{p^2\sin(\frac{n\pi x}{p})}{n^2\pi^2}\bigg|_0^{2p}\right) \\
&= \frac{1}{p}\left(\frac{2p^2}{n\pi}\right) = \frac{2p}{n\pi}.
\end{aligned}
$$

So the Fourier series of f is

$$
p + \sum_{n=1}^{\infty}\frac{2p}{n\pi}\sin\left(\frac{n\pi x}{p}\right) = p + \frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{p}\right).
$$

Remark: This series is equal to $f(x)$ everywhere it is continuous.