

# Linear PDEs and the Principle of Superposition

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Partial Differential Equations  
Lecture 8

# Linear differential operators

**Definition:** A *linear differential operator* (in the variables  $x_1, x_2, \dots, x_n$ ) is a sum of terms of the form

$$A(x_1, x_2, \dots, x_n) \frac{\partial^{a_1+a_2+\dots+a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \dots \partial x_n^{a_n}},$$

where each  $a_i \geq 0$ .

**Examples:** The following are linear differential operators.

1. The Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

2.  $W = c^2 \nabla^2 - \frac{\partial^2}{\partial t^2}$

$$3. H = c^2 \nabla^2 - \frac{\partial}{\partial t}$$

$$4. T = \frac{\partial}{\partial t} - v_1 \frac{\partial}{\partial x_1} - v_2 \frac{\partial}{\partial x_2} - \dots - v_n \frac{\partial}{\partial x_n} = \frac{\partial}{\partial t} - \mathbf{v} \cdot \nabla$$

5. The general first order linear operator (in two variables):

$$D_1 = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} + C(x, y)$$

6. The general second order linear operator (in two variables):

$$D_2 = A(x, y) \frac{\partial^2}{\partial x^2} + 2B(x, y) \frac{\partial^2}{\partial x \partial y} + C(x, y) \frac{\partial^2}{\partial y^2} \\ + D(x, y) \frac{\partial}{\partial x} + E(x, y) \frac{\partial}{\partial y} + F(x, y)$$

## Theorem

*If  $D$  is a linear differential operator (in the variables  $x_1, x_2, \dots, x_n$ ),  $u_1$  and  $u_2$  are functions (in the same variables), and  $c_1$  and  $c_2$  are constants, then*

$$D(c_1 u_1 + c_2 u_2) = c_1 D u_1 + c_2 D u_2.$$

## Remarks:

- This follows immediately from the fact that each partial derivative making up  $L$  has this property, e.g.

$$\frac{\partial^3}{\partial x_1^2 \partial x_2} (c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} + c_2 \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}.$$

- This property extends (in the obvious way) to any number of functions and constants.

# Linear PDEs

**Definition:** A *linear PDE* (in the variables  $x_1, x_2, \dots, x_n$ ) has the form

$$Du = f \tag{1}$$

where:

- $D$  is a linear differential operator (in  $x_1, x_2, \dots, x_n$ ),
- $f$  is a function (of  $x_1, x_2, \dots, x_n$ ).

We say that (1) is *homogeneous* if  $f \equiv 0$ .

# Examples

The following are linear PDEs.

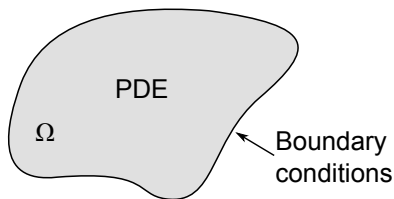
1. The Laplace equation:  $\nabla^2 u = 0$  (homogeneous)
2. The wave equation:  $c^2 \nabla^2 u - \frac{\partial^2 u}{\partial t^2} = 0$  (homogeneous)
3. The heat equation:  $c^2 \nabla^2 u - \frac{\partial u}{\partial t} = 0$  (homogeneous)
4. The Poisson equation:  $\nabla^2 u = f(x_1, x_2, \dots, x_n)$   
(inhomogeneous if  $f \neq 0$ )
5. The advection equation:  $\frac{\partial u}{\partial t} + \kappa \frac{\partial u}{\partial x} + ru = k(x, t)$   
(inhomogeneous if  $k \neq 0$ )
6. The telegraph equation:  $\frac{\partial^2 u}{\partial t^2} + 2B \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} + Au = 0$   
(homogeneous)

# Boundary value problems

A *boundary value problem* (BVP) consists of:

- a domain  $\Omega \subseteq \mathbb{R}^n$ ,
- a PDE (in  $n$  independent variables) to be solved in the interior of  $\Omega$ ,
- a collection of *boundary conditions* (BCs) to be satisfied on the boundary of  $\Omega$ .

The data for a BVP:



# Linear boundary conditions

**Definition:** Let  $\Omega \subseteq \mathbb{R}^n$  be the domain of a BVP and let  $A$  be a subset of the boundary of  $\Omega$ .

We say that a BC on  $A$  is *linear* if it has the form

$$\delta u|_A = f|_A \tag{2}$$

where:

- $\delta$  is a linear differential operator (in  $x_1, x_2, \dots, x_n$ ),
- $f$  is a function (of  $x_1, x_2, \dots, x_n$ ).

(The notation  $\cdot|_A$  means “restricted to  $A$ .”) We say that (2) is *homogeneous* if  $f \equiv 0$ .



# Examples

The following are linear BCs.

1. *Dirichlet conditions:*  $u|_A = f|_A$ , such as

$$u(x, 0) = f(x) \text{ for } 0 < x < L, \text{ or } u(L, t) = 0 \text{ for } t > 0$$

2. *Neumann conditions:*  $\left. \frac{\partial u}{\partial \mathbf{n}} \right|_A = f|_A$ , where  $\frac{\partial u}{\partial \mathbf{n}}$  is the directional derivative perpendicular to  $A$ , such as

$$u_t(x, 0) = g(x) \text{ for } 0 < x < L, \text{ or } u_x(0, t) = 0 \text{ for } t > 0$$

3. *Robin conditions:*  $u + a \left. \frac{\partial u}{\partial \mathbf{n}} \right|_A = f|_A$ , such as

$$u(L, t) + u_x(L, t) = 0 \text{ for } t > 0$$

# The principle of superposition

## Theorem

*Let  $D$  and  $\delta$  be linear differential operators (in the variables  $x_1, x_2, \dots, x_n$ ), let  $f_1$  and  $f_2$  be functions (in the same variables), and let  $c_1$  and  $c_2$  be constants.*

- If  $u_1$  solves the linear PDE  $Du = f_1$  and  $u_2$  solves  $Du = f_2$ , then  $u = c_1u_1 + c_2u_2$  solves  $Du = c_1f_1 + c_2f_2$ . In particular, if  $u_1$  and  $u_2$  both solve the same homogeneous linear PDE, so does  $u = c_1u_1 + c_2u_2$ .*
- If  $u_1$  satisfies the linear BC  $\delta u|_A = f_1|_A$  and  $u_2$  satisfies  $\delta u|_A = f_2|_A$ , then  $u = c_1u_1 + c_2u_2$  satisfies  $\delta u|_A = c_1f_1 + c_2f_2|_A$ . In particular, if  $u_1$  and  $u_2$  both satisfy the same homogeneous linear BC, so does  $u = c_1u_1 + c_2u_2$ .*

## Remarks on the superposition principle

- It is an easy consequence of the linearity of  $D, \delta$ , e.g. if  $Du_1 = f_1$  and  $Du_2 = f_2$ , then

$$D(c_1 u_1 + c_2 u_2) = c_1 Du_1 + c_2 Du_2 = c_1 f_1 + c_2 f_2.$$

- It extends (in the obvious way) to any number of functions and constants.
- It implies that linear combinations of functions that satisfy homogeneous linear PDEs/BCs satisfy *the same equations*.

## Non-example

**Warning:** The principle of superposition can *easily* fail for nonlinear PDEs or boundary conditions.

Consider the nonlinear PDE

$$u_x + u^2 u_y = 0.$$

One solution of this PDE is

$$u_1(x, y) = \frac{-1 + \sqrt{1 + 4xy}}{2x}.$$

However, the function  $u = cu_1$  *does not* solve the same PDE unless  $c = 0, \pm 1$ .

# Superposition and separation of variables

Consider a linear BVP consisting of the following data:

- (A) A *homogeneous* linear PDE on a region  $\Omega \subseteq \mathbb{R}^n$ ;
- (B) A (finite) list of *homogeneous* linear BCs on (part of)  $\partial\Omega$ ;
- (C) A (finite) list of *inhomogeneous* linear BCs on (part of)  $\partial\Omega$ .

Roughly speaking, to solve such a problem one:

1. Finds all “separated” solutions to (A) and (B).
  - This amounts to solving a collection of linear ODE BVPs linked by separation constants.
  - Superposition guarantees *any linear combination* of separated solutions also solves (A) and (B).
2. Determines the specific linear combination of separated solutions that solves (C).

## Remarks on separation of variables

- When separated solutions involve sines and cosines, finding the solutions to inhomogeneous BCs utilize Fourier series/half-range expansions.
- More generally, one must make use of “Fourier like” series involving other families of orthogonal functions (e.g. Sturm-Liouville theory).
- When there are *no* homogeneous BCs, or “too many” inhomogeneous BCs, one can “homogenize” parts of the problem and then superimpose these partial results to get the complete solution.
- Depending on the shape of the domain in question, successful separation of variables may require change of coordinates.