The One-Dimensional Heat Equation

R. C. Daileda

Trinity University

Partial Differential Equations
Lecture 9

Introduction
The heat equation

Goal: Model heat (thermal energy) flow in a one-dimensional object (thin rod).

Set up: Place rod along x-axis, and let

$$u(x,t) = \text{ temperature in rod at position } x, \text{ time } t.$$

Under ideal conditions (e.g. perfect insulation, no external heat sources, uniform rod material), one can show the temperature must satisfy

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \quad \left(\begin{array}{c}\text{the } \textit{one-dimensional} \\ \textit{heat equation}\end{array}\right)$$

The constant c^2 is called the *thermal diffusivity* of the rod.

Initial and Boundary Conditions

We now assume the rod has finite length L and lies along the interval $[0, L]$. To completely determine u we must also specify:

Initial conditions: The initial temperature profile

$$u(x, 0) = f(x) \text{ for } 0 < x < L.$$

Boundary conditions: Specific behavior at $x_0 \in \{0, L\}$:

1. Constant temperature: $u(x_0, t) = T$ for $t > 0$.
2. Insulated end: $u_x(x_0, t) = 0$ for $t > 0$.
3. Radiating end: $u_x(x_0, t) = Au(x_0, t)$ for $t > 0$.

Solving the Heat Equation

Case 1: homogeneous Dirichlet boundary conditions

We now apply separation of variables to the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L,\ \ t > 0), \\ u(0,t) &= u(L,t) = 0 && (t > 0), \\ u(x,0) &= f(x) && (0 < x < L). \end{aligned}$$

We seek separated solutions of the form $u(x,t) = X(x)T(t)$. In this case

$$\left.\begin{aligned} u_t &= XT' \\ u_{xx} &= X''T \end{aligned}\right\} \Rightarrow \ XT' = c^2 X'' T \ \Rightarrow \ \frac{X''}{X} = \frac{T'}{c^2 T} = k.$$

Together with the boundary conditions we obtain the system

$$\begin{aligned} &X'' - kX = 0,\ \ X(0) = X(L) = 0, \\ &T' - c^2 kT = 0. \end{aligned}$$

Already know: up to constant multiples, the only solutions to the BVP in X are

$$k = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2,$$
$$X = X_n = \sin(\mu_n x) = \sin\left(\frac{n\pi x}{L}\right), \;\; n \in \mathbb{N}.$$

Therefore T must satisfy

$$T' - c^2 kT = T' + \underbrace{\left(\frac{cn\pi}{L}\right)}_{\lambda_n}{}^2 T = 0$$

$$T' = -\lambda_n^2 T \;\;\Rightarrow\;\; T = T_n = b_n e^{-\lambda_n^2 t}.$$

We thus have the *normal modes* of the heat equation:

$$u_n(x,t) = X_n(x)T_n(t) = b_n e^{-\lambda_n^2 t}\sin(\mu_n x), \;\; n \in \mathbb{N}.$$

Superposition and initial condition

Applying the principle of superposition gives the general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(\mu_n x).$$

If we now impose our initial condition we find that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

which is the sine series expansion of $f(x)$. Hence

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Remarks

- As before, if the sine series of $f(x)$ is already known, solution can be built by simply including exponential factors.
- One can show that this is the *only* solution to the heat equation with the given initial condition.
- Because of the decaying exponential factors:
  - The normal modes tend to zero (exponentially) as $t \to \infty$.
  - Overall, $u(x,t) \to 0$ (exponentially) *uniformly in* x as $t \to \infty$.
  - As c increases, $u(x,t) \to 0$ more rapidly.

This agrees with intuition.

Example

Solve the heat problem

$$
\begin{aligned}
u_t &= 3u_{xx} && (0 < x < 2,\quad t > 0),\\
u(0,t) &= u(2,t) = 0 && (t > 0),\\
u(x,0) &= 50 && (0 < x < 2).
\end{aligned}
$$

We have $c = \sqrt{3}$, $L = 2$ and, by exercise 2.3.1 (with $p = L = 2$)

$$f(x) = 50 = \frac{200}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Since $\lambda_{2k+1} = \dfrac{c(2k+1)\pi}{L} = \dfrac{\sqrt{3}(2k+1)\pi}{2}$, we obtain

$$u(x,t) = \frac{200}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-3(2k+1)^2\pi^2 t/4}\sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Solving the Heat Equation

Case 2a: steady state solutions

Definition: We say that $u(x,t)$ is a *steady state solution* if $u_t \equiv 0$ (i.e. u is time-independent).

If $u(x,t) = u(x)$ is a steady state solution to the heat equation then

$$u_t \equiv 0 \;\Rightarrow\; c^2 u_{xx} = u_t = 0 \;\Rightarrow\; u_{xx} = 0 \;\Rightarrow\; u = Ax + B.$$

Steady state solutions can help us deal with inhomogeneous Dirichlet boundary conditions. Note that

$$\left.\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} B &= T_1 \\ AL + B &= T_2 \end{aligned}\right\} \Rightarrow u = \left(\frac{T_2 - T_1}{L}\right)x + T_1.$$

Solving the Heat Equation

Case 2b: inhomogeneous Dirichlet boundary conditions

Now consider the heat problem

$$\begin{aligned} &u_t = c^2 u_{xx} && (0 < x < L, \;\; t > 0), \\ &u(0,t) = T_1, \;\; u(L,t) = T_2 && (t > 0), \\ &u(x,0) = f(x) && (0 < x < L). \end{aligned}$$

Step 1: Let u_1 denote the steady state solution from above:

$$u_1 = \left(\frac{T_2 - T_1}{L}\right) x + T_1.$$

Step 2: Let $u_2 = u - u_1$.

Remark: By superposition, u_2 still solves the heat equation.

The boundary and initial conditions satisfied by u_2 are

$$\begin{aligned} u_2(0,t) &= u(0,t) - u_1(0) = T_1 - T_1 = 0, \\ u_2(L,t) &= u(L,t) - u_1(L) = T_2 - T_2 = 0, \\ u_2(x,0) &= f(x) - u_1(x). \end{aligned}$$

Step 3: Solve the heat equation with homogeneous Dirichlet boundary conditions and initial conditions above. This yields u_2.

Step 4: Assemble $u(x,t) = u_1(x) + u_2(x,t)$.

Remark: According to our earlier work, $\lim_{t\to\infty} u_2(x,t) = 0$.

- We call $u_2(x,t)$ the *transient* portion of the solution.
- We have $u(x,t) \to u_1(x)$ as $t \to \infty$, i.e. the solution tends to the steady state.

Example

Solve the heat problem.

$$\begin{aligned} &u_t = 3u_{xx} && (0 < x < 2,\ \ t > 0),\\ &u(0,t) = 100, \quad u(2,t) = 0 && (t > 0),\\ &u(x,0) = 50 && (0 < x < 2). \end{aligned}$$

We have $c = \sqrt{3}$, $L = 2$, $T_1 = 100$, $T_2 = 0$ and $f(x) = 50$. The steady state solution is

$$u_1 = \left(\frac{0-100}{2}\right)x + 100 = 100 - 50x.$$

The corresponding homogeneous problem for u_2 is thus

$$\begin{aligned} &u_t = 3u_{xx} && (0 < x < 2,\ \ t > 0),\\ &u(0,t) = u(2,t) = 0 && (t > 0),\\ &u(x,0) = 50 - (100 - 50x) \quad = 50(x-1) && (0 < x < 2). \end{aligned}$$

According to exercise 2.3.7 (with $p = L = 2$), the sine series for $50(x-1)$ is

$$\frac{-100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{2k\pi x}{2}\right),$$

i.e. only *even* modes occur. Since $\lambda_{2k} = \dfrac{c2k\pi}{L} = \sqrt{3}k\pi$,

$$u_2(x,t) = \frac{-100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-3k^2\pi^2 t}\sin\left(k\pi x\right).$$

Hence

$$u(x,t) = u_1(x) + u_2(x,t) = 100 - 50x - \frac{100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-3k^2\pi^2 t}\sin\left(k\pi x\right).$$

Solving the Heat Equation
Case 3: homogeneous Neumann boundary conditions

Let's now consider the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L \ , \ 0 < t), \\ u_x(0,t) &= u_x(L,t) = 0 && (0 < t), \\ u(x,0) &= f(x) && (0 < x < L), \end{aligned}$$

in which we assume the ends of the rod are *insulated*.

As before, assuming $u(x,t) = X(x)T(t)$ yields the system

$$\begin{aligned} & X'' - kX = 0, \ \ X'(0) = X'(L) = 0, \\ & T' - c^2 kT = 0. \end{aligned}$$

Note that the boundary conditions on X *are not the same* as in the Dirichlet condition case.

Solving for X

Case 1: $k = \mu^2 > 0$. We need to solve $X'' - \mu^2 X = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \;\Rightarrow\; r = \pm\mu,$$

which gives the general solution $X = c_1 e^{\mu x} + c_2 e^{-\mu x}$. The boundary conditions tell us that

$$0 = X'(0) = \mu c_1 - \mu c_2, \;\; 0 = X'(L) = \mu c_1 e^{\mu L} - \mu c_2 e^{-\mu L},$$

or in matrix form

$$\begin{pmatrix} \mu & -\mu \\ \mu e^{\mu L} & -\mu e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant is $\mu^2(e^{\mu L} - e^{-\mu L}) \neq 0$, we must have $c_1 = c_2 = 0$, and so $X \equiv 0$.

Case 2: $k = 0$. We need to solve $X'' = 0$. Integrating twice gives

$$X = c_1 x + c_2.$$

The boundary conditions give $0 = X'(0) = X'(L) = c_1$. Taking $c_2 = 1$ we get the solution

$$X = X_0 = 1.$$

Case 3: $k = -\mu^2 < 0$. We need to solve $X'' + \mu^2 X = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r = \pm i\mu,$$

which gives the general solution $X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions yield

$$\begin{aligned} 0 &= X'(0) = -\mu c_1 \sin 0 + \mu c_2 \cos 0 = \mu c_2 \quad \Rightarrow \quad c_2 = 0, \\ 0 &= X'(L) = -\mu c_1 \sin(\mu L) + \mu c_2 \cos(\mu L) = -\mu c_1 \sin(\mu L). \end{aligned}$$

In order to have $X \not\equiv 0$, this shows that we need

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu L = n\pi \quad \Rightarrow \quad \mu = \mu_n = \frac{n\pi}{L} \quad (n \in \mathbb{Z}).$$

Taking $c_1 = 1$ we obtain

$$X = X_n = \cos(\mu_n x) \quad (n \in \mathbb{N}).$$

Remarks:

- We only need $n > 0$, since cosine is an even function.
- When $n = 0$ we get $X_0 = \cos 0 = 1$, which agrees with the $k = 0$ result.

Normal modes and superposition

As before, for $k = -\mu_n^2$, we obtain $T = T_n = a_n e^{-\lambda_n^2 t}$.

We therefore have the normal modes

$$u_n(x,t) = X_n(x)T_n(t) = a_n e^{-\lambda_n^2 t}\cos(\mu_n x) \quad (n \in \mathbb{N}_0),$$

where $\mu_n = n\pi/L$ and $\lambda_n = c\mu_n$.

The principle of superposition now gives the general solution

$$u(x,t) = u_0 + \sum_{n=1}^{\infty} u_n = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos(\mu_n x)$$

to the heat equation with (homogeneous) Neumann boundary conditions.

Initial conditions

If we now impose our initial condition we find that

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},$$

which is simply the $2L$-periodic cosine expansion of $f(x)$. Hence

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad (n \in \mathbb{N}).$$

Remarks:

- As before, if the cosine series of $f(x)$ is already known, $u(x,t)$ can be built by simply including exponential factors.
- Because of the exponential factors, $\lim_{t\to\infty} u(x,t) = a_0$, which is the *average initial temperature*.

Example

Solve the following heat problem:

$$
\begin{aligned}
u_t &= \frac{1}{4} u_{xx}, && 0 < x < 1 \ , \ 0 < t, \\
u_x(0,t) &= u_x(1,t) = 0, && 0 < t, \\
u(x,0) &= 100x(1-x), && 0 < x < 1.
\end{aligned}
$$

We have $c = 1/2$, $L = 1$ and $f(x) = 100x(1-x)$. Therefore

$$a_0 = \int_0^1 100x(1-x)\,dx = \frac{50}{3}$$

$$a_n = 2\int_0^1 100x(1-x)\cos n\pi x\,dx = \frac{-200(1+(-1)^n)}{n^2\pi^2}, \;\; n \geq 1.$$

Since $\lambda_n = cn\pi/L = n\pi/2$, plugging everything into the general solution we get

$$u(x,t) = \frac{50}{3} - \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(1+(-1)^n)}{n^2}e^{-n^2\pi^2 t/4}\cos n\pi x.$$

As in the case of Dirichlet boundary conditions, the exponential terms decay rapidly with t. We therefore have

$$\lim_{t\to\infty} u(x,t) = \frac{50}{3}.$$

Deriving the heat equation

(Ideal) Assumptions:

- Rod is perfectly insulated with negligible thickness, i.e. heat only moves horizontally.
- No external heat sources or sinks.
- Rod material is uniform, i.e. has constant *specific heat*, s, and (linear) mass density, ρ.

Recall that

$$s = \begin{cases} \text{amount of heat required to raise one unit} \\ \text{of mass by one unit of temperature.} \end{cases}$$

Consider a small segment of the rod at position x of length Δx.

The thermal energy in this segment at time t is

$$E(x, x + \Delta x, t) \approx u(x, t) s\rho \Delta x.$$

Fourier's law of heat conduction states that the (rightward) heat flux at any point is

$$-K_0 u_x(x, t),$$

where K_0 is the *thermal conductivity* of the rod material.

Remark: Fourier's law quantifies the notion that thermal energy moves from hot to cold.

Appealing to the law of conservation of energy,

$$\underbrace{\frac{\partial}{\partial t}(u(x,t)s\rho\Delta x)}_{\text{heat flux through segment}} \approx \underbrace{-K_0 u_x(x,t)}_{\text{heat flux in at left end}} + \underbrace{K_0 u_x(x+\Delta x,t)}_{\text{heat flux in at right end}},$$

or

$$u_t(x,t) \approx \frac{K_0}{s\rho}\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}.$$

Letting $\Delta x \to 0$ improves the approximation and leads to the *one-dimensional heat equation*

$$u_t = c^2 u_{xx},$$

where $c^2 = \frac{K_0}{s\rho}$ is called the *thermal diffusivity*.