Exercise 1. Textbook exercise 1.3.

Exercise 2. Let $n > 2$ be an integer and suppose that the n terms of the arithmetic progression

$$p,\ p+d,\ p+2d,\ p+3d, \ldots,\ p+(n-1)d \tag{1}$$

are prime. Let $q < n$ be prime. We will prove that $q|d$.[1]

- **a.** Assume $q \nmid d$. Show that no two of the first q terms of the progression (1)

$$p,\ p+d,\ p+2d, \ldots,\ p+(q-1)d \tag{2}$$

  are congruent modulo q. [*Suggestion:* Argue by contradiction.]
- **b.** Use part **a** to show that one of the terms, say $p + td$, in (2) is divisible by q.
- **c.** Show that if $p < n$, then one of the terms in (1) would be composite. Hence $n \leq p$.[2]
- **d.** Use parts **b** and **c** to show that $q < p+td$ and hence that $p+td$ is a composite member of (1). This conclusion establishes the result.

Remark The Green-Tao theorem, proven in 2004, asserts that there exist arbitrarily long arithmetic progressions of primes. In other words, for any $n \geq 2$ one can find $d \geq 2$ so that (1) consists of only prime numbers. Even so, it wasn't until 2015 that the longest arithmetic progression of primes, of length only 26, was found by Bryan Little:

$$161004359399459161 + t \cdot \underbrace{47715109 \cdot \prod_{p \leq 23} p}_{d}, \quad 0 \leq n \leq 25.$$

Notice that d is indeed divisible by every prime less than 26, as we just proved it must be.

[1] Note that this puts a severe restriction on the structure of *consecutive* primes in an arithmetic progression.

[2] This puts yet another restriction on the progression (1): an n term progression must start with a prime $p \geq n$. This also proves an arithmetic progression of primes must be finite: its length cannot exceed its first member.