Number Theory I | Assignment 7.2
Spring 2018 | Due February 28

Exercise 1. Let $\alpha : R \to S$ be an isomorphism of rings.

a. Prove that $\alpha(1_R) = 1_S$. [*Suggestion:* The identity in a ring is unique. So it suffices to show $\alpha(1_R) \cdot s = s \cdot \alpha(1_R) = s$ for all $s \in S$. To do this use the surjectivity of α.]

b. Given $r \in R$, prove that $r \in R^\times$ if and only if $\alpha(r) \in S^\times$. This shows α maps $R^\times$ bijectively onto $S^\times$.

Exercise 2. Find $\varphi(n)$ for each value of n below.

a. 2592

b. 4851

c. 111111

d. 15!

Exercise 3. Prove the following generalization of the multiplicative property of φ: for $m, n \in \mathbb{N}$, if $d = (m, n)$, then

$$\varphi(mn)\varphi(d) = \varphi(m)\varphi(n)d.$$

[*Suggestion:* Use the formula $\varphi(a) = a \prod_{p|a}(1 - p^{-1})$.]

Exercise 4. Let $m, n \in \mathbb{N}$ with $m|n$. Prove that $\varphi(mn) = m\varphi(n)$. [*Suggestion:* See the suggestion for the previous exercise and show that $p|mn$ if and only if $p|n$.]