Euler Equations

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1 Change of Variables

An *Euler equation* is a homogeneous linear second order ODE of the form

$$x^2y'' + axy' + by = 0, \quad x \neq 0. \tag{1}$$

In order to find the general solution, we perform the change of variable[1]

$$t = \log|x|.$$

According to the chain rule

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} \quad \Rightarrow \quad xy' = \frac{dy}{dt}.$$

Differentiating both sides of this final equation with respect to x and using the chain rule again gives us

$$y' + xy'' = \frac{d}{dx}\frac{dy}{dt} = \frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{d^2y}{dt^2}\frac{1}{x} \quad \Rightarrow \quad xy' + x^2y'' = \frac{d^2y}{dt^2} \quad \Rightarrow \quad x^2y'' = \frac{d^2y}{dt^2} - xy' = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

From these computations it follows that our original ODE (1) becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + a\frac{dy}{dt} + by = 0 \quad \Leftrightarrow \quad \frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0. \tag{2}$$

Notice that (2) is linear and homogeneous with constant coefficients and therefore can be solved by the standard technique involving the roots of its characteristic equation.

2 The Indicial Equation

We call the characteristic equation

$$r^2 + (a-1)r + b = 0 \tag{3}$$

of (2) the *indicial equation* of (1). We will now use its roots to describe the solutions to (1).

Case 1. (3) has two distinct real roots, $r_1 \neq r_2$. In this case the solutions to (2) are given by

$$y = c_1e^{r_1t} + c_2e^{r_2t}.$$

[1] As usual in mathematical circles, log denotes the natural logarithm.

Back substituting $t = \log|x|$ this becomes

$$\boxed{y = c_1|x|^{r_1} + c_2|x|^{r_2}}.$$

Case 2. (3) has one repeated real root r_1. In this situation we know the general solution to (2) to be

$$y = e^{r_1 t}(c_1 + c_2 t).$$

Back substitution now gives

$$\boxed{y = |x|^{r_1}(c_1 + c_2 \log|x|)}.$$

Case 3. (3) has conjugate complex roots $\alpha \pm i\beta$ with $\beta \neq 0$. Now the general solution of (2) is

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)).$$

Upon back substitution we arrive at the general solution to (1):

$$\boxed{y = |x|^{\alpha}(c_1 \cos(\beta \log|x|) + c_2 \sin(\beta \log|x|))}.$$

3 Summary

We have proven the following result.

Theorem 1. *The general solution to the Euler equation*

$$x^2 y'' + axy' + by = 0, \quad x \neq 0,$$

depends on the roots r_1, r_2 of its indicial equation

$$r^2 + (a-1)r + b = 0$$

as follows.

1. *If r_1, r_2 are real and distinct (i.e. $(a-1)^2 - 4b > 0$), then*

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}.$$

2. *If $r_1 = r_2$ (i.e. $(a-1)^2 - 4b = 0$), then*

$$y = |x|^{r_1}(c_1 + c_2 \log|x|).$$

3. *If $r_1, r_2 = \alpha \pm i\beta$ with $\beta \neq 0$ (i.e. $(a-1)^2 - 4b < 0$), then*

$$y = |x|^{\alpha}(c_1 \cos(\beta \log|x|) + c_2 \sin(\beta \log|x|)).$$

As a final remark we note that if we restrict the domain to $x > 0$, the absolute values $|\cdot|$ become unnecessary.