Math 3357 Spring 2018

Partial Differential Equations

First Exam

Solutions

1. Solve the initial value problem

$$\frac{\partial u}{\partial x} + (x - yx)\frac{\partial u}{\partial y} = 0, \quad (x, y) \in (0, \infty) \times \mathbb{R},$$
$$u(0, y) = \sin(y).$$

Solution 1: Because the PDE has the form

$$\frac{\partial u}{\partial x} + P(x, y)\frac{\partial u}{\partial y} = 0,$$

we may use the "naïve" method of characteristics. In this case we begin by solving

$$\frac{dy}{dx} = x - yx, \tag{1}$$

which is both linear and separable. We will demonstrate how to solve it as either type.

First treating (1) as linear we rewrite it as

$$\frac{dy}{dx} + xy = x \tag{2}$$

and multiply by the integrating factor

$$\mu(x) = \exp\left(\int x\,dx\right) = e^{x^2/2}$$

which reduces (2) to

$$\frac{d}{dx}e^{x^2/2}y = xe^{x^2/2}.$$

Integrating both sides we obtain

$$\int \frac{d}{dx}e^{x^2/2}y\,dx = \int xe^{x^2/2}\,dx \;\Rightarrow\; e^{x^2/2}y = e^{x^2/2} + C \;\Rightarrow\; y = 1 + Ce^{-x^2/2}$$
$$\Rightarrow\; C = e^{x^2/2}(y-1).$$

If we instead treat the characteristic ODE as separable we find that (the constant C is not necessarily the same at every occurrence)

$$\frac{dy}{1-y} = x\,dx \;\Rightarrow\; \int \frac{dy}{1-y} = \int x\,dx \;\Rightarrow\; -\log|1-y| = \frac{x^2}{2} + C \;\Rightarrow$$
$$\frac{1}{|1-y|} = Ce^{x^2/2}\;(C > 0) \;\Rightarrow\; \frac{1}{1-y} = Ce^{x^2/2}\;(C \neq 0) \;\Rightarrow\; C = e^{x^2/2}(y-1),$$

in agreement with the earlier computation.[1]

Hence, in either case, we find that the general solution to the PDE is

$$u(x,y) = f(e^{x^2/2}(y-1)).$$

Imposing the initial condition on this solution we find that

$$\sin(y) = u(0,y) = f(y-1) \quad \Rightarrow \quad \sin(y+1) = f((y+1)-1) = f(y)$$

so that the particular solution we seek is

$$\boxed{u(x,y) = \sin(e^{x^2/2}(y-1)+1)}.$$

Solution 2: Being quasi-linear, this PDE is automatically amenable to the "full strength" version of the method of characteristics. We parametrize the initial curve as

$$x_0(a) = 0, \;\; y_0(a) = a, \;\; z_0(a) = \sin(a)$$

and use this to construct the system of characteristic ODEs:

$$\begin{array}{lll} \dfrac{dx}{ds} = 1, & \dfrac{dy}{ds} = x - yx, & \dfrac{dz}{ds} = 0, \\ x(0) = 0, & y(0) = a, & z(0) = \sin(a). \end{array}$$

The final equation implies that z is constant so that the initial condition $z(0) = \sin(a)$ implies $z = \sin(a)$. Integrating the first equation gives $x = s + C$. The initial condition tells us that $0 = x(0) = 0 + C$ so that $C = 0$ and $x = s$. This reduces th middle equation to

$$\frac{dy}{ds} = s - ys,$$

which we solved above, obtaining $C = e^{s^2/2}(y-1)$. Before we solve for y we set $s = 0$ and impose the initial condition: $C = e^0(y(0)-1) = a - 1$. Putting this back in the expression for y and adding 1 to both sides yields $a = 1 + e^{s^2/2}(y-1)$. Since $x = s$ we further obtain $a = 1 + e^{x^2/2}(y-1)$ and hence the solution is

$$\boxed{u(x,y) = z = \sin(a) = \sin(1 + e^{x^2/2}(y-1)).}$$

[1] It is tempting to stop at the end of the first line, set $C = \frac{x^2}{2} + \log|1-y|$ and take the general solution to be $u(x,y) = f\left(\frac{x^2}{2} + \log|1-y|\right)$, but this isn't sufficient for our purposes: such a solution isn't defined when $y = 1$ whereas we are requiring that our solution be defined for all $y \in \mathbb{R}$. Even if we were to accept this as our general solution, because of the absolute value, solving for f in the initial value condition $\sin(y) = f(\log|1-y|)$ is somewhat subtle.

2. Solve the initial value problem

$$\begin{aligned} &2x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u, \quad (x,y) \in \mathbb{R} \times (0,\infty), \\ &u(x,0) = 3x. \end{aligned}$$

Solution: Because u doesn't appear in the coefficients on the LHS of the PDE, technically we could apply the modified naïve method of characteristics to this problem. The solution by this method, however, is rather elaborate. So we simply use the ordinary method of characteristics. The initial curve is

$$x_0(a) = a, \quad y_0(a) = 0, \quad z_0(a) = 3z,$$

so that the system of characteristic ODEs is

$$\begin{array}{lll} \dfrac{dx}{ds} = 2x, & \dfrac{dy}{ds} = 1, & \dfrac{dz}{ds} = z, \\ x(0) = a, & y(0) = 0, & z(0) = 3a. \end{array}$$

The first and final equations are exponential growth equations with solutions

$$x = ae^{2s}, \quad z = 3ae^{s}.$$

Ans, yes, that's all the work you were expected to show. Integrating the middle equation yields $y = s + C$ which, when we impose $y(0) = 0$, becomes $y = s$. Solving for a in the equation for x tells us that $a = xe^{-2s}$ so that $z = 3ae^s = 3xe^{-2s}e^s = 3xe^{-s} = 3xe^{-y}$, since $s = y$. Hence the solution to the initial value problem is

$$\boxed{u(x,y) = 3xe^{-y}.}$$

3. When making the linear change of variables

$$\begin{aligned} \alpha &= ax + by, \\ \beta &= cx + dy, \\ ad - bc &\neq 0, \end{aligned}$$

we have seen that the chain rule for a function $u(x, y)$ leads to the formulae

$$\begin{aligned} u_{xx} &= a^2 u_{\alpha\alpha} + 2ac u_{\alpha\beta} + c^2 u_{\beta\beta}, \\ u_{yy} &= b^2 u_{\alpha\alpha} + 2bd u_{\alpha\beta} + d^2 u_{\beta\beta}, \\ u_{xy} &= ab u_{\alpha\alpha} + (ad + bc) u_{\alpha\beta} + cd u_{\beta\beta}. \end{aligned}$$

a. Find values of a, b, c and d so that the linear change of variables above reduces the PDE

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0 \tag{3}$$

to

$$u_{\alpha\beta} = 0.$$

Solution: Under the given substitution the PDE becomes

$$(a^2 - 5ab + 6b^2)u_{\alpha\alpha} + (2ac + 5(ad + bc) + 12bd)u_{\alpha\beta} + (c^2 - 5cd + 6d^2)u_{\beta\beta} = 0.$$

Notice that

$$a^2 - 5ab + 6b^2 = (a - 2b)(a - 3b) = 0 \iff a = 2b \text{ or } a = 3b.$$

So if we take $\boxed{a = 2}$ and $\boxed{b = 1}$, the first of these equations is satisfied and the $u_{\alpha\alpha}$ term drops out of the PDE. The $u_{\beta\beta}$ coefficient is identical with the $u_{\alpha\alpha}$ coefficient, just with c and d replacing a and b. We therefore have the same conditions for it to vanish: $c = 2d$ or $c = 3d$. In order not to violate $ad - bc \neq 0$ we must use the second equation, and $\boxed{c = 3}$, $\boxed{d = 1}$ are convenient choices.[2]

It only remains to be sure that the $u_{\alpha\beta}$ coefficient is nonzero with these choices. It's not hard to show that if $a = 2$, $b = 1$, $c = 3$ and $d = 1$, then $2ac + 5(ad + bc) + 12bd = -1$ so that our original PDE has now become

$$-u_{\alpha\beta} = 0 \iff u_{\alpha\beta} = 0$$

as desired.

[2] These are not the only "correct" values of a, b, c and d. So long as $a = 2b$ and $c = 3d$, or $a = 3d$ and $c = 2d$ we will have a valid substitution that produces the desired result.

b. Use part **a** to show that the general solution to (3) is given by

$$u(x, y) = F(2x + y) + G(3x + y).$$

Solution: Using the change of variables $\alpha = 2x + y$, $\beta = 3x + y$ found in part **a**, we know that the PDE becomes

$$u_{\alpha\beta} = 0.$$

So we simply integrate twice:

$$\begin{aligned} u_\alpha &= \int u_{\alpha\beta}\, \partial\beta = \int 0\, \partial\beta = f(\alpha), \\ u &= \int u_\alpha\, \partial\alpha = \int f(\alpha)\, \partial\alpha = F(\alpha) + G(\beta), \end{aligned}$$

where f is an arbitrary differentiable function, F is its antiderivative and $G(\beta)$ is the "constant" of integration with respect to α. Back substituting for α and β we obtain

$$\boxed{u(x, y) = F(2x + y) + G(3x + y).}$$

4. Recall *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

a. Show that $u(x,y) = \dfrac{x}{x^2+y^2}$ is a solution to Laplace's equation.

Solution: We simply compute the necessary second partial derivatives.

$$\begin{aligned}
u_x &= \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\
u_{xx} &= \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} = \frac{-2x(x^2+y^2)\left(x^2+y^2+2(y^2-x^2)\right)}{(x^2+y^2)^4} \\
&= \frac{-2x(3y^2-x^2)}{(x^2+y^2)^3} \\
u_y &= \frac{-2xy}{(x^2+y^2)^2} \\
u_{yy} &= \frac{(x^2+y^2)^2(-2x) + 2xy(2)(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{2x(x^2+y^2)(-(x^2+y^2)+4y^2)}{(x^2+y^2)^4} \\
&= \frac{2x(3y^2-x^2)}{(x^2+y^2)^3}
\end{aligned}$$

It is now immediately apparent that $u_{xx} + u_{yy} = 0$.

b. By symmetry and linearity, it immediately follows that $v(x,y) = \dfrac{-y}{x^2+y^2}$ also solves Laplace's equation. Show that u and v are related through the *Cauchy-Riemann equations*

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.
\end{aligned}$$

Solution: We already have the partial derivatives of u from part **a**, so we just compute those of v:

$$v_x = \frac{2xy}{(x^2+y^2)^2} = -u_y \quad \text{from part } \mathbf{a},$$

$$v_y = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = u_x \quad \text{also by part } \mathbf{a}.$$

5. Let

$$\begin{aligned} f_1(x) &= e^{-x} \\ f_2(x) &= (2x-1)e^{-x} \\ f_3(x) &= (2x^2-4x+1)e^{-x} \end{aligned}$$

a. Show that the functions f_1, f_2 and f_3 are pairwise orthogonal on the interval $[0, \infty)$. You may find it useful to know that

$$\int_0^\infty x^n e^{-2x}\,dx = \frac{n!}{2^{n+1}} \quad \text{for} \quad n \in \mathbb{N}_0.$$

Solution: We have

$$\begin{aligned} \langle f_1, f_2\rangle &= \int_0^\infty f_1(x)f_2(x)\,dx = \int_0^\infty (2x-1)e^{-2x}\,dx \\ &= 2\int_0^\infty xe^{-2x}\,dx - \int_0^\infty e^{-2x}\,dx \\ &= 2\cdot\frac{1!}{2^2} - \frac{0!}{2} = \frac{1}{2} - \frac{1}{2} = 0, \end{aligned}$$

$$\begin{aligned} \langle f_1, f_3\rangle &= \int_0^\infty f_1(x)f_3(x)\,dx = \int_0^\infty (2x^2-4x+1)e^{-2x}\,dx \\ &= 2\int_0^\infty x^2e^{-2x}\,dx - 4\int_0^\infty xe^{-2x}\,dx + \int_0^\infty e^{-2x}\,dx \\ &= 2\cdot\frac{2!}{2^3} - 4\cdot\frac{1!}{2^2} + \frac{0!}{2} = \frac{1}{2} - 1 + \frac{1}{2} = 0, \end{aligned}$$

$$\begin{aligned} \langle f_2, f_3\rangle &= \int_0^\infty f_2(x)f_3(x)\,dx = \int_0^\infty (4x^3-10x^2+6x-1)e^{-2x}\,dx \\ &= 4\int_0^\infty x^3e^{-2x}\,dx - 10\int_0^\infty x^2e^{-2x}\,dx + 6\int_0^\infty xe^{-2x}\,dx - \int_0^\infty e^{-2x}\,dx \\ &= 4\cdot\frac{3!}{2^4} - 10\cdot\frac{2!}{2^3} + 6\cdot\frac{1!}{2^2} - \frac{0!}{2} = \frac{3}{2} - \frac{5}{2} + \frac{3}{2} - \frac{1}{2} = 0. \end{aligned}$$

b. *Use inner products* to express x^2e^{-x} as a linear combination of f_1, f_2 and f_3.

Solution: Let $g(x) = x^2e^{-x}$. We have

$$\langle g, f_1\rangle = \int_0^\infty g(x)f_1(x)\,dx = \int_0^\infty x^2e^{-2x}\,dx = \frac{2!}{2^3} = \frac{1}{4},$$

$$\langle f_1, f_1\rangle = \int_0^\infty f_1(x)f_1(x)\,dx = \int_0^\infty e^{-2x}\,dx = \frac{0!}{2} = \frac{1}{2},$$

$$\begin{aligned}\langle g, f_2\rangle &= \int_0^\infty g(x)f_2(x)\,dx = \int_0^\infty (2x^3 - x^2)e^{-2x}\,dx \\ &= 2\int_0^\infty x^3e^{-2x}\,dx - \int_0^\infty x^2e^{-2x}\,dx = 2\cdot\frac{3!}{2^4} - \frac{2!}{2^3} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}\langle f_2, f_2\rangle &= \int_0^\infty f_2(x)f_2(x)\,dx = \int_0^\infty (4x^2 - 4x + 1)e^{-2x}\,dx \\ &= 4\int_0^\infty x^2e^{-2x}\,dx - 4\int_0^\infty xe^{-2x}\,dx + \int_0^\infty e^{-2x}\,dx \\ &= 4\cdot\frac{2!}{2^3} - 4\cdot\frac{1!}{2^2} + \frac{0!}{2} = 1 - 1 + \frac{1}{2} = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}\langle g, f_3\rangle &= \int_0^\infty g(x)f_3(x)\,dx = \int_0^\infty (2x^4 - 4x^3 + x^2)e^{-2x}\,dx \\ &= 2\int_0^\infty x^4e^{-2x}\,dx - 4\int_0^\infty x^3e^{-2x}\,dx + \int_0^\infty x^2e^{-2x}\,dx \\ &= 2\cdot\frac{4!}{2^5} - 4\cdot\frac{3!}{2^4} + \frac{2!}{2^3} = \frac{3}{2} - \frac{3}{2} + \frac{1}{4} = \frac{1}{4},\end{aligned}$$

$$\begin{aligned}\langle f_3, f_3\rangle &= \int_0^\infty f_3(x)f_3(x)\,dx = \int_0^\infty (4x^4 - 16x^3 + 20x^2 - 8x + 1)e^{-2x}\,dx \\ &= 4\int_0^\infty x^4e^{-2x}\,dx - 16\int_0^\infty x^3e^{-2x}\,dx + 20\int_0^\infty x^2e^{-2x}\,dx - 8\int_0^\infty xe^{-2x}\,dx + \int_0^\infty e^{-2x}\,dx \\ &= 4\cdot\frac{4!}{2^5} - 16\cdot\frac{3!}{2^4} + 20\cdot\frac{2!}{2^3} - 8\cdot\frac{1!}{2^2} + \frac{0!}{2} = 3 - 6 + 5 - 2 + \frac{1}{2} = \frac{1}{2}\end{aligned}$$

Hence the Fourier coefficients are

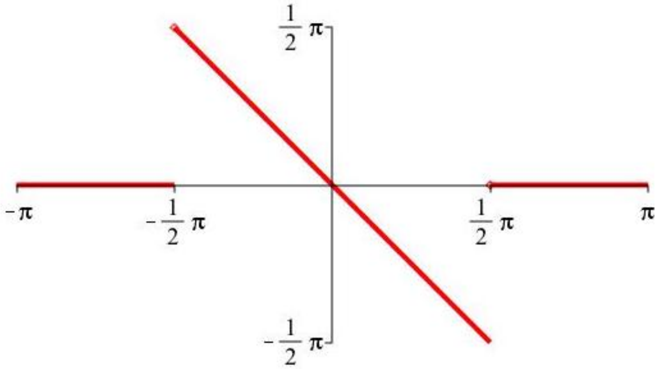
$$a_1 = \frac{\langle g, f_1\rangle}{\langle f_1, f_1\rangle} = \frac{1/4}{1/2} = \frac{1}{2},\quad a_2 = \frac{\langle g, f_2\rangle}{\langle f_2, f_2\rangle} = \frac{1/2}{1/2} = 1,\quad a_3 = \frac{\langle g, f_3\rangle}{\langle f_3, f_3\rangle} = \frac{1/4}{1/2} = \frac{1}{2},$$

so that

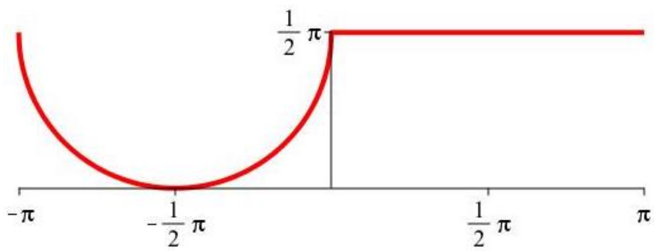
$$\boxed{g = \frac{1}{2}f_1 + f_2 + \frac{1}{2}f_3.}$$

6. A single period of the graph of a 2π-periodic function f is shown. Carefully sketch three periods of the graph of f and three periods of the graph of the Fourier series of f.

a.



b.



c.