Math 3357 Spring 2018

Partial Differential Equations

Second Exam

Solutions

1. Find the Fourier series of the $2p$-periodic saw-tooth wave given by $f(x) = 2p - x$ for $0 < x < 2p$.

Solution. This function is neither odd nor even, however $g(x) = f(x) - p$ is easily seen to be an odd function (draw a picture). Its Fourier coefficients are therefore given by $a_n = 0$ for all n and, using tabular integration by parts

$$\begin{aligned} b_n &= \frac{1}{p}\int_0^{2p} (p-x)\sin\left(\frac{n\pi x}{p}\right)\,dx \\ &= \frac{1}{p}\left(-\frac{p}{n\pi}(p-x)\cos\left(\frac{n\pi x}{p}\right) - \frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right)\Bigg|_0^{2p} \\ &= \frac{1}{p}\left(\frac{p^2}{n\pi}\cos(2n\pi) + \frac{p^2}{n\pi}\right) = \frac{2p}{n\pi} \end{aligned}$$

Therefore

$$f(x) = p + g(x) = \boxed{p + \frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{p}\right).}$$

Remark. One might also notice that translating the function in Example 2.2.1 up by $\pi/2$, dilating it along the y-axis by a factor of $2p/\pi$ and along the x-axis by a factor of p/π, one obtains $f(x)$, and so one can obtain the Fourier series for $f(x)$ by performing these operations on the Fourier series derived in that example.

2. Find the cosine expansion of the function $g(x) = x^2$, $0 < x < 1$.

Solution. The cosine coefficients are given by

$$a_0 = \frac{1}{1}\int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}$$

and for $n \geq 1$, using tabular integration by parts,

$$\begin{aligned}
a_n &= \frac{2}{1}\int_0^1 x^2 \cos(n\pi x)\, dx \\
&= 2\left(\frac{x^2}{n\pi}\sin(n\pi x) + \frac{2x}{n^2\pi^2}\cos(n\pi x) - \frac{2}{n^3\pi^3}\sin(n\pi x)\right)\Bigg|_0^1 \\
&= \frac{4}{n^2\pi^2}\cos(n\pi) = \frac{(-1)^n 4}{n^2\pi^2}.
\end{aligned}$$

Hence the cosine series is

$$\boxed{\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x).}$$

Remark. One can also obtain the solution by realizing that the even 2-periodic extension of x^2 is the function of Exercise 2.3.4 with $p = 1$ and using the series given there.

3. Use the method of separation of variables to replace the partial differential equation

$$xu_{xx} + u_t = 0$$

by a pair of ordinary differential equations. Do not attempt to solve the resulting equations.

Solution. Write $u(x,t) = X(x)T(t) = XT$. Substituting this into the PDE gives us

$$xX''T + XT' = 0 \;\Rightarrow\; xX''T = -XT' \;\Rightarrow\; \frac{xX''}{X} = -\frac{T'}{T} = k$$

since the left-hand side is a function of x only, whereas the right-hand side is a function of t only. Cross multiplying and moving all terms to one side in each equation gives us the ODE pair

$$\boxed{xX'' - kX = 0, \;\; T' + kT = 0.}$$

4. A perfectly elastic string of length 3 moves according to the equation $u_{tt} = 2u_{xx}$. It is stretched into the shape of the graph of the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 2, \\ -2(x-3) & \text{if } 2 \leq x \leq 3. \end{cases}$$

and released. Write down a formula that gives the shape of the string at any later time.

Solution. We appeal to the solution to the one dimensional wave equation given on page 119 of our textbook. We have $L = 3$ and $c = \sqrt{2}$. Since the initial velocity is zero, $b_n^* = 0$ for all n. Moreover, since b_n is simply the nth sine coefficient of the initial shape, and the sine expansion of the initial shape is given by

$$\frac{18}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2}\sin\left(\frac{n\pi x}{3}\right)$$

according to Exercise 2.4.17 (with $h = a = 2$ and $p = 3$), we have

$$b_n = \frac{18 \sin\left(\frac{2n\pi}{3}\right)}{\pi^2 n^2}.$$

Since $\lambda_n = \sqrt{2}n\pi/3$, the motion of the string is therefore described by

$$\boxed{u(x,t) = \frac{18}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2} \sin\left(\frac{n\pi x}{3}\right) \cos\left(\frac{\sqrt{2}n\pi t}{3}\right).}$$

5. An ideal laterally insulated conducting rod of length 5 is initially heated so that the temperature throughout is given by $f(x) = 20(4 - x)$. Its ends are then insulated as well. Give an expression for the temperature in the rod at any later time. What happens to the temperature in the long run?

Solution. According to Example 3.6.1, the temperature is given by including appropriate exponential factors in the cosine series for $f(x)$. The cosine coefficients of $f(x)$ are given by

$$a_0 = \frac{1}{5}\int_0^5 20(4 - x)\,dx = 4\left(4x - \frac{x^2}{2}\right)\Big|_0^5 = 4\left(20 - \frac{25}{2}\right) = 30$$

and for $n \geq 1$, by tabular integration by parts,

$$\begin{aligned} a_n = \frac{2}{5}\int_0^5 20(4 - x)\cos\left(\frac{n\pi x}{5}\right)dx &= 8\left(\frac{5}{n\pi}(4 - x)\sin\left(\frac{n\pi x}{5}\right) - \frac{25}{n^2\pi^2}\cos\left(\frac{n\pi x}{5}\right)\right)\Big|_0^5 \\ &= -\frac{200}{n^2\pi^2}\cos(n\pi) + \frac{200}{n^2\pi^2} \\ &= \frac{200(1 + (-1)^{n+1})}{n^2\pi^2}. \end{aligned}$$

Since $\lambda_n = cn\pi/5$, $e^{-\lambda_n^2 t} = e^{-c^2n^2\pi^2t/25}$ so that the temperature is given by

$$\boxed{30 + \frac{200}{\pi^2}\sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n^2} e^{-c^2n^2\pi^2t/25} \cos\left(\frac{n\pi x}{5}\right).}$$

Since the exponential terms tend rapidly to zero as $t \to \infty$, we see that the temperature will tend to 30 in the long run.

6. Consider the $2p$-periodic triangle wave given by $f(x) = |x|$ for $-p < x < p$. It's Fourier series can be shown to be

$$\frac{p}{2} - \frac{4p}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left(\frac{(2k+1)\pi x}{p}\right)$$

(you do not need to verify this). Use this information to obtain the Fourier series for the $2p$-periodic triangle wave given by

$$g(x) = \begin{cases} x & \text{if } -p/2 \leq x < p/2, \\ p - x & \text{if } p/2 \leq x < 3p/2. \end{cases}$$

You may find it useful to know that $\cos(A+B) = \cos A \cos B - \sin A \sin B$. [*Suggestion:* Draw pictures of the two waves to determine their relationship.]

Solution. One way to obtain $g(x)$ from $f(x)$ is to translate the graph of the latter down and to the left, both by $p/2$ units. That is,

$$g(x) = f\left(x + \frac{p}{2}\right) - \frac{p}{2}.$$

This means the Fourier series for $g(x)$ is

$$\begin{aligned}
-\frac{p}{2}+\frac{p}{2} - \frac{4p}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos\left(\frac{(2k+1)\pi(x+p/2)}{p}\right) \\
&= -\frac{4p}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos\left(\frac{(2k+1)\pi x}{p} + \frac{(2k+1)\pi}{2}\right) \\
&= -\frac{4p}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\left(\cos\left(\frac{(2k+1)\pi x}{p}\right)\cos\left(\frac{(2k+1)\pi}{2}\right) - \sin\left(\frac{(2k+1)\pi x}{p}\right)\sin\left(\frac{(2k+1)\pi}{2}\right)\right) \\
&= -\frac{4p}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\left(\cos\left(\frac{(2k+1)\pi x}{p}\right)\cdot 0 - \sin\left(\frac{(2k+1)\pi x}{p}\right)(-1)^k\right) \\
&= \boxed{\frac{4p}{\pi^2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{(2k+1)\pi x}{p}\right)}.
\end{aligned}$$