Math 3357 Spring 2018

Partial Differential Equations

Third Exam

Solutions

1. The Kronecker delta function[1] $\delta(x, y)$ represents a point impulse at the origin and has the property that

$$\iint_\Omega \delta(x-a, y-b) f(x, y)\, dA = f(a, b) \tag{1}$$

for any region $\Omega \subseteq \mathbb{R}^2$ and any $(a, b) \in \Omega$. Suppose we have an ideal 3×5 elastic membrane, with $c = 1/\pi$, at equilibrium. If we impart a downward point impulse velocity at $(1, 1)$, write down an expression that describes the resulting motion of the membrane. [*Suggestion:* Translate the delta function so that its impulse occurs at $(1, 1)$ and use (1).]

Solution. We use the solution of the 2-D wave equation on a rectangle provided in the third bullet point of the equation sheet. We have $a =$ and $b = 5$. Since the membrane begins at equilibrium, $f(x, y) = 0$ so that $B_{mn} = 0$ for all m, n. Furthermore, an initial downward point impulse velocity is given by $g(x, y) = -\delta(x-1, y-1)$. Hence

$$B^*_{mn} = \frac{4}{15\lambda_{mn}} \int_0^b \int_0^a -\delta(x-1, y-1) \sin(\mu_m x) \sin(\nu_n y)\, dx\, dy = -\frac{4}{15\lambda_{mn}} \sin(\mu_m) \sin(\nu_n)$$

where $\mu_m = m\pi/3$, $\nu_n = n\pi/5$. Since $c = 1/\pi$,

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_2} = \sqrt{\frac{m^2}{9} + \frac{n^2}{25}} \;\Rightarrow\; 15\lambda_{mn} = \sqrt{25m^2 + 9n^2}.$$

So

$$B^*_{mn} = -\frac{4 \sin(\mu_m) \sin(\nu_n)}{\sqrt{25m^2 + 9n^2}}.$$

Therefore

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} -\frac{4 \sin(\mu_m) \sin(\nu_n)}{\sqrt{25m^2 + 9n^2}} \sin\left(\sqrt{\frac{m^2}{9} + \frac{n^2}{25}}\, t\right) \sin\left(\frac{m\pi x}{3}\right) \sin\left(\frac{n\pi y}{5}\right).}$$

[1] Technically δ is not a function but a *distribution*, a subtlety we will ignore.

2. A thin $a \times b$ metal plate with insulated faces has the temperature along each edge held constant. If the constant temperatures are T_1, T_2, T_3 and T_4, starting on the bottom and going clockwise, find the resulting steady-state temperature distribution throughout the plate.

Solution. The $2L$-periodic half-range sine expansion of the constant function T has coefficients given by

$$b_n = \frac{2}{L}\int_0^L T \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2T}{L}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Big|_0^L = \frac{2T}{n\pi}\left((-1)^{n+1}+1\right).$$

So if we apply the solution from the fourth bullet point of the formula sheet (the Dirichlet problem on a rectangle) with $f_1 = T_1$, $f_2 = T_3$, $g_1 = T_2$ and $g_2 = T_4$ we find that

$$\begin{aligned}
A_n &= \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \cdot \frac{2T_1}{n\pi}\left((-1)^{n+1}+1\right) = \frac{2T_1\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}, \\
B_n &= \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \cdot \frac{2T_3}{n\pi}\left((-1)^{n+1}+1\right) = \frac{2T_3\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}, \\
C_n &= \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \cdot \frac{2T_2}{n\pi}\left((-1)^{n+1}+1\right) = \frac{2T_2\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}, \\
D_n &= \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \cdot \frac{2T_4}{n\pi}\left((-1)^{n+1}+1\right) = \frac{2T_4\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}.
\end{aligned}$$

The temperature is therefore given by

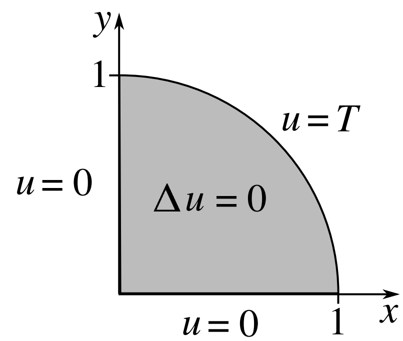
$$\boxed{\begin{aligned}
u(x,y) = &\sum_{n=1}^{\infty} \frac{2T_1\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \\
&+ \sum_{n=1}^{\infty} \frac{2T_3\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \\
&+ \sum_{n=1}^{\infty} \frac{2T_2\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi(a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \\
&+ \sum_{n=1}^{\infty} \frac{2T_4\left((-1)^{n+1}+1\right)}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right).
\end{aligned}}$$

3. Consider the Dirichlet problem on the quarter of the unit disk in the first quadrant given in polar coordinates by

$$\Delta u = 0, \;\; 0 < r < 1, \;\; 0 < \theta < \frac{\pi}{2},$$

$$u(r,0) = u(r,\pi/2) = 0, \;\; 0 < r < 1,$$

$$u(1,\theta) = T, \;\; 0 < \theta < \frac{\pi}{2}.$$



a. Use separation of variables to reduce the homogeneous components of this problem to a pair of ODE boundary value problems. [*Remark:* There is an implicit boundary condition at the origin.]

Solution. In polar coordinates the Laplace equation is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Assuming $u(r,\theta) = R(r)\Theta(\theta)$ and plugging into the above we obtain

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \;\Rightarrow\; \frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0 \;\Rightarrow\; \frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = k,$$

where in the second step we have multiplied both sides by $r^2/R\Theta$. Clearing denominators gives us the ODEs

$$\boxed{\begin{aligned} r^2R'' + rR' - kR &= 0, \\ \Theta'' + k\Theta &= 0. \end{aligned}}$$

The two homogeneous boundary conditions tell us that

$$\begin{aligned} 0 = R(r)\Theta(0) \;&\Rightarrow\; \boxed{\Theta(0) = 0}, \\ 0 = R(r)\Theta(\pi/2) \;&\Rightarrow\; \boxed{\Theta(\pi/2) = 0}, \end{aligned}$$

since we do not want R to be identically zero. Finally, we want our solution to remain bounded throughout the quarter disk. I particular, it should remain finite as we approach the origin (though any angle). This requires:

$$\boxed{R(0+) \text{ is finite.}}$$

b. Solve the boundary value problem in θ found in **a**. [*Remark:* It should be a familiar problem. Feel free to simply cite the solution.]

Solution. We encountered the boundary value problem

$$\begin{aligned} &\Theta'' + k\Theta = 0, \\ &\Theta(0) = \Theta(\pi/2) = 0, \end{aligned}$$

in the context of the vibrating string of length $\pi/2$ (and elsewhere). Up to constant multiples, its solutions are

$$\boxed{\Theta = \Theta_n = \sin\left(\frac{n\pi\theta}{\pi/2}\right) = \sin(2n\theta), \quad n \in \mathbb{N}}$$

with $\boxed{k = \mu_n^2 = (n\pi/(\pi/2))^2 = 4n^2}$.

c. Use the result of part **b** to solve the boundary value problem in r found in **a**. [*Remark:* It should be an Euler equation.]

Solution. For $n \in \mathbb{N}$, taking $k = 4n^2$ in the equation for R gives us the Euler equation

$$r^2 R'' + rR' - 4n^2 R = 0$$

with indicial equation

$$\rho^2 + (1-1)\rho - 4n^2 = \rho^2 - 4n^2 = 0 \quad \rho = \pm 2n.$$

Therefore $R = c_1 r^{2n} + c_2 r^{-2n}$. Since $r^{-2n} \to \infty$ as $r \to 0^+$, $R(0+)$ will only be finite (as required) if $c_2 = 0$. So, up to a constant multiple, R is given by

$$\boxed{R = R_n = r^{2n}, \quad n \in \mathbb{N}}.$$

d. Use superposition to express the general solution as an infinite linear combination of the modes found in parts **b** and **c**.

Solution. According to the previous two steps, the normal modes of this problem are

$$u_n(r,\theta) = r^{2n}\sin(2n\theta), \quad n \in \mathbb{N}.$$

Superposition gives the general solution

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} c_n u_n(r,\theta) = \sum_{n=1}^{\infty} c_n r^{2n}\sin(2n\theta)}$$

to the (homogeneous) Laplace equation and homogeneous boundary conditions on the straight edges of the domain.

e. Impose the final (inhomogeneous) boundary condition and express the coefficients of the series solution in part **d** in terms of T (integral expressions are not sufficient).

Solution. Setting $r = 1$ in the general solution and imposing the boundary condition there yields

$$T = u(1,\theta) = \sum_{n=1}^{\infty} c_n \sin(2n\theta)$$

which is simply the π-periodic half-range sine expansion of the constant function T. According to the preceding exercise we have

$$c_n = \frac{2T}{n\pi}\left((-1)^{n+1}+1\right) \quad \Rightarrow \quad \boxed{u(r,\theta) = \sum_{n=1}^{\infty} \frac{2T}{n\pi}\left((-1)^{n+1}+1\right) r^{2n}\sin(2n\theta)}.$$

4. Consider the second order ODE

$$(1-x)y'' + xy' - y = 0. \tag{2}$$

a. Show that $a = 0$ is an ordinary point of (2).

Solution. First we put (2) into standard form:

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = 0.$$

Using the sum of the geometric series we have

$$\frac{x}{1-x} = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n \quad \text{for } |x| < 1,$$

$$-\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} -x^n \quad \text{for } |x| < 1,$$

so that both coefficient coefficient functions are given by power series centered at $a = 0$ with positive radii of convergence, i.e. are analytic at $a = 0$. Hence $a = 0$ is an ordinary point of the ODE (2).

b. Give a lower bound for the radii of convergence of every power series solution to (2) centered at $a = 0$.

Solution. According to part **a**, the coefficient functions have radii of convergence equal to 1, hence every solution analytic at $a = 0$ has radius at least 1.

c. Find the recursion relation satisfied by the coefficients of every power series solution to (2) centered at $a = 0$.

Solution. Writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and plugging into the ODE (2) gives us

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 - a_0) + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n\right)x^n = 0.$$

According to the identity principle we must have

$$2a_2 - a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = \frac{a_0}{2}},$$

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n = 0$$

$$\Rightarrow \quad \boxed{a_{n+2} = \frac{(n+1)na_{n+1} - (n-1)a_n}{(n+2)(n+1)}, \quad n \geq 1}.$$

One can easily check that the second equation actually subsumes the first when $n = 0$.

d. Use part **c** to find explicit power series expressions for two linearly independent solutions to (2).

Solution. Taking $a_0 = 1$ and $a_1 = 0$ we find that

$$a_2 = \frac{1}{2}, \quad a_3 = \frac{2a_2}{3 \cdot 2} = \frac{1}{3 \cdot 2}, \quad a_4 = \frac{3 \cdot 2a_3 - a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3 \cdot 2}, \quad a_5 = \frac{4 \cdot 3a_4 - 2a_3}{5 \cdot 4} = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2}, \ldots$$

$$\Rightarrow a_n = \frac{1}{n!} \quad \text{for } n \geq 2$$

$$\Rightarrow \boxed{y_1 = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!}}.$$

With $a_0 = 0$ and $a_1 = 1$ we instead obtain

$$a_2 = \frac{0}{2} = 0, \quad a_3 = \frac{2a_2}{3 \cdot 2} = 0, \quad a_4 = \frac{3 \cdot 2a_3 - a_2}{4 \cdot 3} = 0, \quad a_5 = \frac{4 \cdot 3a_4 - 2a_3}{5 \cdot 4} = 0, \ldots$$

$$\Rightarrow a_n = 0 \quad \text{for } n \geq 2$$

$$\Rightarrow \boxed{y_2 = x}.$$

e. Identify the power series you found in **d** as familiar functions.

Solution. Clearly there's nothing to do with y_2. The function y_1 is just e^x without its linear term. That is

$$\boxed{y_1 = e^x - x}.$$

5. Consider the second order ODE

$$x^2y'' + (x^2 + \frac{1}{4})y = 0. \tag{3}$$

a. Show that $a = 0$ is a regular singular point of (3).

Solution. In standard form the ODE (3) becomes

$$y'' + \left(1 + \frac{1}{4x^2}\right) y = 0,$$

which has $p(x) = 0$ and $q(x) = 1 + \frac{1}{4x^2}$. Since $q(x)$ is *not* analytic at $a = 0$ (it isn't even defined there), but $xp(x) = 0$ and $x^2q(x) = x^2 + \frac{1}{4}$ are both (finite) power series centered at $a = 0$, with infinite radius of convergence, $a = 0$ is a regular singular point of (3).

b. Find the values of r for which (3) has a Frobenius solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

Give a lower bound on the radius of convergence of the power series factor.

Solution. Since $xp(x)$ and $x^2q(x)$ from part **a** were shown to have infinite radius of convergence, the power series factor in every Frobenius solution will have infinite radius of convergence as well.

To determine the value(s) of r for which a Frobenius solution exists, we must solve the indicial equation. Since

$$p_0 = \lim_{x\to 0} xp(x) = 0,$$
$$q_0 = \lim_{x\to 0} x^2q(x) = \frac{1}{4},$$

the indicial equation is

$$r^2 + (0-1)r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2 = 0$$

with the single root

$$\boxed{r = \frac{1}{2}}.$$

c. Find the recursion relation satisfied by the coefficients in each Frobenius solution.

Solution. Writing

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

and substituting into (3) we obtain

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+x^2\sum_{n=0}^{\infty}a_nx^{n+r}+\frac{1}{4}\sum_{n=0}^{\infty}a_nx^{n+r}=0$$
$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}+\frac{1}{4}\sum_{n=0}^{\infty}a_nx^{n+r}=0$$
$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}+\frac{1}{4}\sum_{n=0}^{\infty}a_nx^{n+r}=0$$
$$\left(r(r-1)+\frac{1}{4}\right)a_0x^r+\left(r(r+1)+\frac{1}{4}\right)a_1x^{r+1}$$
$$+\sum_{n=2}^{\infty}\left((n+r)(n+r-1)a_n+a_{n-2}+\frac{1}{4}a_n\right)x^{n+r}=0$$

Equating the coefficients in the final expression to zero yields (since $a_0\neq 0$)

$$\left(r(r-1)+\frac{1}{4}\right)a_0=0 \;\Rightarrow\; r^2-r+\frac{1}{4}=0 \;\Rightarrow\; r=\frac{1}{2},$$
$$\left(r(r+1)+\frac{1}{4}\right)a_1=0 \;\Rightarrow\; \left(\frac{1}{2}\cdot\frac{3}{2}+\frac{1}{4}\right)a_1=0 \;\Rightarrow\; \boxed{a_1=0},$$
$$(n+r)(n+r-1)a_n+a_{n-2}+\frac{1}{4}a_n=0 \;\Rightarrow\; \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)a_n+\frac{1}{4}a_n+a_{n-2}=0$$
$$\Rightarrow\; n^2a_n+a_{n-2}=0 \;\Rightarrow\; \boxed{a_n=\frac{-a_{n-2}}{n^2}\text{ for } n\geq 2}.$$

d. By taking $a_0=1$, find explicit expressions for the power series factors of each Frobenius solution.

Solution. Since $a_1=0$, the recursion relation tells us that

$$a_3=\frac{-a_1}{9}=0 \;\Rightarrow\; a_5=\frac{-a_3}{25}=0 \;\Rightarrow\; a_7=\frac{-a_5}{49}=0 \;\Rightarrow\; \cdots \;\Rightarrow\; a_{2k+1}=0 \text{ for } k\geq 0.$$

Moreover, with $a_0=1$ we have

$$a_2=\frac{-a_0}{2^2}=\frac{-1}{2^2} \;\Rightarrow\; a_4=\frac{-a_2}{4^2}=\frac{1}{2^24^2} \;\Rightarrow\; a_6=\frac{-a_4}{6^2}=\frac{-1}{2^24^26^2} \;\Rightarrow\; \cdots$$
$$\Rightarrow\; a_{2k}=\frac{(-1)^k}{2^24^26^2\cdots(2k)^2}=\frac{(-1)^k}{(2\cdot4\cdot6\cdots(2k))^2}=\frac{(-1)^k}{2^{2k}(k!)^2} \text{ for } k\geq 0.$$

Hence our only Frobenius solution is

$$\boxed{y=x^{1/2}\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{2^{2k}(k!)^2}}.$$

- The two-dimensional Laplace operator, in Cartesian (resp. polar) coordinates, is

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

- The solution of the boundary value problem

$$\begin{aligned} &u_t = c^2 \Delta u, \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ &u(0, y, t) = u(a, y, t) = 0, \quad 0 \le y \le b, \quad t \ge 0, \\ &u(x, 0, t) = u(x, b, t) = 0, \quad 0 \le x \le a, \quad t \ge 0, \\ &u(x, y, 0) = f(x, y), \quad 0 < x < a, \quad 0 < y < b, \end{aligned}$$

is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(\mu_m x) \sin(\nu_n y) e^{-\lambda_{mn}^2 t},$$

where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, and

$$A_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin(\mu_m x) \sin(\nu_n y) \, dx \, dy.$$

- The solution of the boundary value problem

$$\begin{aligned} &u_{tt} = c^2 \Delta u, \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ &u(0, y, t) = u(a, y, t) = 0, \quad 0 \le y \le b, \quad t \ge 0, \\ &u(x, 0, t) = u(x, b, t) = 0, \quad 0 \le x \le a, \quad t \ge 0, \\ &u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y), \quad 0 < x < a, \quad 0 < y < b, \end{aligned}$$

is

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(B_{mn} \cos\left(\lambda_{mn} t\right) + B_{mn}^* \sin\left(\lambda_{mn} t\right) \right) \sin(\mu_m x) \sin(\nu_n y),$$

where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, and

$$\begin{aligned} B_{mn} &= \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin(\mu_m x) \sin(\nu_n y) \, dx \, dy, \\ B_{mn}^* &= \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x, y) \sin(\mu_m x) \sin(\nu_n y) \, dx \, dy. \end{aligned}$$

- The solution of the boundary value problem

$$\begin{aligned} &\Delta u = 0, \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ &u(x, 0) = f_1(x), \quad u(x, b) = f_2(x), \quad 0 \le x \le a, \\ &u(0, y) = g_1(y), \quad u(a, y) = g_2(y), \quad 0 \le x \le b, \end{aligned}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\mu_n x)\sinh(\mu_n(b-y)) + \sum_{n=1}^{\infty} B_n \sin(\mu_n x)\sinh(\mu_n y)$$
$$+ \sum_{n=1}^{\infty} C_n \sinh(\nu_n(a-x))\sin(\nu_n y) + \sum_{n=1}^{\infty} D_n \sinh(\nu_n x)\sin(\nu_n y),$$

where $\mu_n = \frac{n\pi}{a}$, $\nu_n = \frac{n\pi}{b}$ and

$$A_n = \frac{2}{a\sinh(\mu_n b)} \int_0^a f_1(x)\sin(\mu_n x)\,dx, \quad B_n = \frac{2}{a\sinh(\mu_n b)} \int_0^a f_2(x)\sin(\mu_n x)\,dx,$$
$$C_n = \frac{2}{b\sinh(\nu_n a)} \int_0^b g_1(y)\sin(\nu_n y)\,dy, \quad D_n = \frac{2}{b\sinh(\nu_n a)} \int_0^b g_2(y)\sin(\nu_n y)\,dy.$$

- The general solution of the polar coordinate boundary value problem

$$\Delta u = 0, \quad 0 < r < a, \quad 0 < \theta < 2\pi,$$
$$u(a,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi,$$

is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)),$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \quad (n > 0),$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta \quad (n > 0).$$

- The general solution of the Euler equation

$$x^2y'' + axy' + by = 0, \quad x > 0,$$

depends on the roots r_1, r_2 of the indicial equation

$$r^2 + (a-1)r + b = 0$$

as follows:

Case 1. If $r_1 \neq r_2$ are real, then

$$y = c_1x^{r_1} + c_2x^{r_2}.$$

Case 2. If $r_1 = r_2$, then

$$y = c_1x^{r_1} + c_2x^{r_1}\ln x.$$

Case 3. If $r_1, r_2 = \alpha \pm \beta i$ with $\beta \neq 0$, then

$$y = x^{\alpha}(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)).$$

- Power series expansions of a few familiar functions.

PS Expansion	**Valid for**
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	all x
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all x
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	all x
$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	all x
$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	all x
$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$\lvert x\rvert < 1$
$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$\lvert x\rvert < 1$
$(1+x)^k = \sum_{k=0}^{\infty} \binom{k}{n} x^n$	$\lvert x\rvert < 1$

In the final series, $k \in \mathbb{R}$ and the *generalized binomial coefficient* is given by

$$\binom{k}{n} = \frac{k(k-1)(k-2)(k-3)\cdots(k-(n-1))}{n!}.$$

- Trigonometric identities:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\sin A \cos B = \frac{(\sin(A-B) + \sin(A+B))}{2}$$

$$\sin A \sin B = \frac{(\cos(A-B) - \cos(A+B))}{2}$$

$$\cos A \cos B = \frac{(\cos(A-B) + \cos(A+B))}{2}$$