



PARTIAL DIFFERENTIAL EQUATIONS
SPRING 2018

ASSIGNMENT 9
DUE FEBRUARY 22

Exercise 1. Textbook exercise 2.4.4 [*Suggestion:* Obtain the cosine series of f from a horizontal translate of Exercise 2.3.8. The sine series of f will need to be found by integration.]

Exercise 2. Textbook exercise 2.4.5.

Exercise 3. Textbook exercise 3.3.2 [*Suggestions:* Use the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ and use the orthogonality of the trigonometric system.]

Exercise 4. Textbook exercise 3.3.6 [*Suggestions:* Obtain the sine series of g from Exercise 2.3.1. As far as f goes, start with Exercise 2.3.8. Translate it to the left d units, to the right d units, and take the difference. Take $p = 1$, choose c and d appropriately, then translate to the right by 1 unit. This will result in the 2-periodic extension of f . It's a lot less work than it sounds like!]

Exercise 5. Textbook exercise 3.3.7 [*Suggestions:* See the previous suggestion regarding g . To obtain the sine series for f , it may be easier to obtain the cosine series for its derivative then antidifferentiate.]

Exercise 6. Textbook exercise 3.3.9 [*Suggestions:* Integrate a downward translate of the cosine series of Example 2.3.2 to obtain the sine series of f . The function g is already a sine series. Why?]

For a change, you are required to provide the plots requested in each of these problems. Although you may feel free to produce these plots by any means you choose, on the following pages I detail how they can be implemented in Maple.

Consider the Fourier series of Exercise 2.3.8:

$$\frac{cd}{2p} + \frac{2cp}{d\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - \cos\left(\frac{dn\pi}{p}\right) \right) \cos\left(\frac{n\pi x}{p}\right).$$

The first thing we should do is assign actual values to c , d and p , otherwise Maple won't be able to graph anything. We do this as follows:

```
> c:= 2;  
> d:= 3;  
> p:= 4;
```

hitting “enter” after each line. The pair of symbols `:=` is the *assignment operator* and assigns the variable on the left the value on the right.

For convenience I then like to encode the Fourier coefficients into a function as follows:

```
> a := n -> 2*c*p/(d*Pi^2*n^2)*(1-cos(d*n*Pi/p));
```

Here we are assigning a to the function that sends n to $\frac{2cp}{d\pi^2 n^2} \left(1 - \cos\left(\frac{dn\pi}{p}\right) \right)$. We can now use a as we would any function, e.g. we could ask for its value at 5 by typing `a(5)`; . This is the standard way that functions are defined in Maple. It is important to note that the mathematical constant $\pi = 3.141592\dots$ is represented in Maple by the symbol `Pi`, *not* `pi`. The latter is simply an undeclared variable name.

Finally, we build the actual partial sums $s_N(x)$. I like to build the truncation index N into the definition, and this can be accomplished as follows:

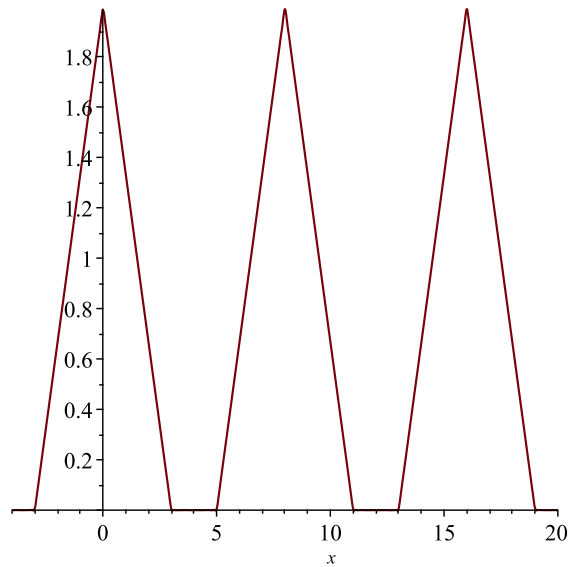
```
> s := (x,N) -> c*d/(2*p) + sum(a(n)*cos(n*Pi*x/p), n=1..N);
```

The first argument to the sum command is the formula for the summands, and the second specifies the index of summation as well as its range. There are two advantages to defining a separately from s_N : it makes coding s_N much simpler (we only had to type `a(n)` and not the entire formula for (n)) and it makes it easier to detect errors since we have broken the code into pieces.

Finally we plot some partial sums. For this we use the `plot` function which, while it has practically unlimited optional arguments, at a bare minimum requires only two: the function to be plotted and the independent variable together with its domain. So, for example, to graph the 50th partial sum of our example over 3 periods from $-p$ to $5p$, we'd simply type

```
> plot(s(x,50),x=-p..5*p);
```

This produces the output



which agrees with the function described in Exercise 2.3.8.

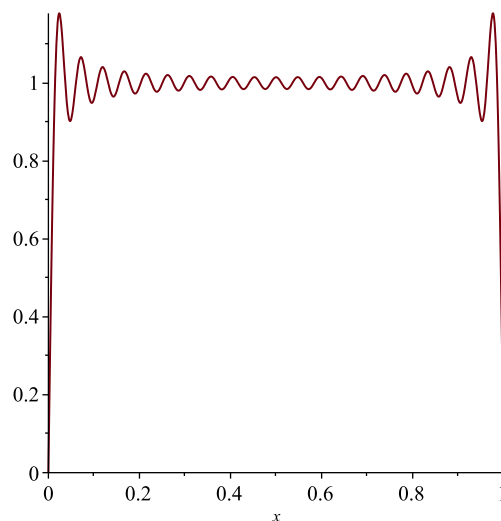
For half-range expansions we do essentially the same thing, except that we only produce a plot over the original domain of the function, since this is where the expansion is the most relevant. The function in Exercise 2.4.1 is only defined for $0 < x < 1$ and its sine series expansion is

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin((2k+1)\pi x).$$

So our Maple code should look something like

```
> b := k -> 4/(Pi*(2*k+1));
> s := (x,N) -> sum(b(k)*sin((2*k+1)*Pi*x),k=0..N);
> plot(s(x,20),x=0..1);
```

This produces the output



which roughly agrees with the function described in Exercise 2.4.1. To improve the accuracy of the approximation we can simply reinsert our cursor in the line with the `plot` command and increase N until we get a picture that's satisfactory. However, we'll never be able to get rid of the "jumps" at the end points due to Gibbs' phenomenon. The best we can do is push them further and further toward the edges.

Finally, let's look at an example of the vibrating string with fixed endpoints. Consider the unit length string with $c = 1/\pi$ of Example 1 from Section 3.3. Its motion is described by the solution

$$u(x, t) = \frac{9}{10\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n^2} \sin(n\pi x) \cos(nt)$$

of the wave equation. To plot this solution we begin as before:

```
> b := n -> 9/(10*Pi^2)*sin(n*Pi/3)/n^2;
> s := (x,t,N) -> sum(b(n)*sin(n*Pi*x)*cos(n*t),n=1..N);
```

If we choose specific values of N (relatively large) and t (arbitrary), and plot $\mathbf{s}(x, \mathbf{t}, N)$ from $x = 0$ to $x = L = 1$, we get snapshots of the motion of the string at different times.

To actually watch the string move, we use the `animate` command. To use it, first we must load the `plots` package:

```
> with(plots);
```

(we only need to do this once per document). The `animate` command is similar to the `plot` command. Here's the syntax to animate the 50th partial sum of the solution for t from 0 to 10:

```
> animate(plot, [s(x,t,50),x=0..1],t=0..10);
```

Now we know that the period of motion of the string is actually $2L/c = 2\pi$, so it makes more sense to animate over the range $t = 0$ to $t = 2\pi$ and loop the result. This can be done by clicking on the plot, then clicking on the drop down menu at the top to the left of the FPS box. Figure 1 shows several frames of the resulting animation

The moral of the story is, aside from actually coding the Fourier coefficients and partial sum functions, when it comes to plotting them its important to know what an appropriate domain is. For Fourier series of generic periodic functions, it's best to plot several periods. For half-range expansions of functions defined on intervals of the form $[0, p]$, one need only plot partial sums over this interval alone. And finally for solutions of the fixed endpoint vibrating string problem, spatially one should only plot over the length of the string: $x = 0$ to $x = L$. And one need only animate over one period of motion, namely $2L/c$, and then loop the result to actually watch the string move indefinitely.

Figure 1: Snapshots of the motion of the string from Example 3.3.1

