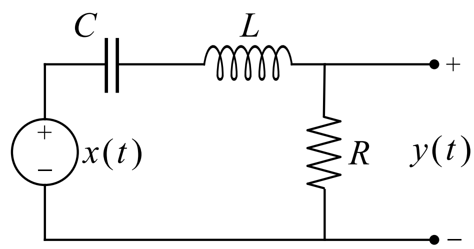
Partial Differential Equations — Project 1: RLC Circuits

Spring 2018 — Due March 2, 5pm

Consider a circuit consisting of a (variable) voltage source, a resistor, an inductor and a capacitor wired in series, as shown below.



This is an example of an *RLC circuit*, and in this project we will investigate the role such a circuit can play in signal processing. We will consider the (time dependent) voltage source $x(t)$ as the input signal to the circuit, and the voltage $y(t)$ across the resistor as the output. The voltage $z(t)$ across the capacitor will serve as an intermediate quantity in our derivations.

If we let $I(t)$ denote the (clockwise) current through the circuit, then the voltages across the resistor, inductor and capacitor are $y(t) = RI(t)$, $LI'(t)$ and $z(t)$, respectively. According to Kirchoff's law, these voltages must add up to the input voltage, so that

$$RI(t) + LI'(t) + z(t) = x(t). \tag{1}$$

If $q(t)$ denotes the charge on the capacitor, then $q(t) = Cz(t)$. Since $I(t) = q'(t)$, this means that $I(t) = Cz'(t)$ and $I'(t) = Cz''(t)$. Substituting these into (1), we find that the input signal $x(t)$ and the voltage $z(t)$ across the capacitor are related by the second order linear differential equation

$$LCz''(t) + RCz'(t) + z(t) = x(t). \tag{2}$$

We will now convert (2) into a differential equation in $y(t)$ instead. Since $y(t) = RI(t) = RCz'(t)$ implies that $z'(t) = y(t)/RC$, differentiating (2) immediately yields

$$LC\frac{y''(t)}{RC} + y'(t) + \frac{y(t)}{RC} = x'(t).$$

Multiplying through by RC finally puts this in the form

$$LCy''(t) + RCy'(t) + y(t) = RCx'(t). \tag{3}$$

We will assume that the input signal is $2p$-periodic, sufficiently smooth, and equal to its Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{p}\right) + b_n \sin\left(\frac{n\pi t}{p}\right)\right). \tag{4}$$

One can easily show that the solutions to the complementary equation

$$LCy''(t) + RCy'(t) + y(t) = 0$$

all decay exponentially as $t \to \infty$, i.e. only contribute transient terms to the output signal $y(t)$. If we furthermore assume that enough time has passed so that these transient contributions have died out, we find that $y(t)$ is also $2p$-periodic, and so has a Fourier series:

$$y(t) = c_0 + \sum_{n=1}^{\infty} \left(c_n \cos\left(\frac{n\pi t}{p}\right) + d_n \sin\left(\frac{n\pi t}{p}\right)\right). \tag{5}$$

Exercises

Exercise 1. By directly substituting the series expressions (4) and (5) into (3), and then comparing the Fourier coefficients of both sides, express c_0, c_n and d_n in terms of a_0, a_n and b_n. You may assume $x(t)$, $y(t)$ and $y'(t)$ are sufficiently smooth for their Fourier series to be differentiated term-wise. You may also find it convenient to introduce the frequency variables $\omega_n = n\pi/p$.

Exercise 2. Create a Maple document which plots (partial sums of) the input and output signals. In order to make your code as portable as possible, it is required that:

- The values for R, L, C and p are stored in variables with these names.
- The coefficients a_n, b_n, c_n and d_n are functions of n, i.e. can be called as `a(n)`, `b(n)`, `c(n)` and `d(n)`.
- The variable `N` is used to specify the length of the partial sums that are plotted.
- Your plots extend from $t = 0$ to $t = 6p$.
- Your code is carefully documented and written so that the various input parameters are easily modified.

Exercise 3. With the values $R = 500$, $L = 0.1$, $C = 1.0 \times 10^{-7}$ and $p = 1/200$, use your code to produce plots for the following input signals.

a. A square wave of amplitude 1

b. A sawtooth wave of amplitude 2

c. A triangle wave of amplitude 5

Exercise 4. The amplitude of a wave of the form $a\cos(\omega t) + b\sin(\omega t)$ is given by $A = \sqrt{a^2 + b^2}$. Use the results of Exercise 1 to show that the amplitude A'_n of the nth summand of the output series (5) is related to the amplitude A_n of the nth summand of the input series (4) by the equation

$$A'_n = \frac{RC\omega_n A_n}{\sqrt{(1 - LC\omega_n^2)^2 + R^2C^2\omega_n^2}}, \tag{6}$$

where $\omega_n = n\pi/p$.

Exercise 5. For the values of R, L and C given in Exercise 3, plot the function

$$f(\omega) = \frac{RC\omega}{\sqrt{(1 - LC\omega^2)^2 + R^2C^2\omega^2}}$$

over an appropriate domain. In light of the relationship (6), explain why this type of RLC circuit is sometimes used in applications as a *bandpass filter*. Where does the maximum of $f(\omega)$ occur in general? What is the maximum value? How is this connected to the "filtering" property of the RLC circuit?

Instructions

- You may work in a group of up to 4 students. Each student in a group is expected to contribute equally to the project. If you feel that one of your group members is not doing his share, please let me know.
- Your written solutions to the exercises must be typed and submitted as a PDF file. Diagrams produced using Maple must be included in the body of this document. The file you submit should be named using the convention `Project1_`*lastname(s)*`.pdf`.
- Your Maple code must be submitted as a separate worksheet file, named using the convention `Project1_`*lastname(s)*`.mw`.
- Your PDF and Maple documents must be uploaded to `http://tlearn.trinity.edu` by 5pm on March 3. Please only upload one copy of each document per group. Late projects *will not be accepted.*
- Failure to adhere to these guidelines will be penalized. If you have any questions or concerns, please ask me.