Hyperbolic Functions and Solutions to Second Order ODEs

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1 Hyperbolic Functions

For any x, the *hyperbolic cosine* and *hyperbolic sine* of x are defined to be

$$\cosh x = \frac{e^x + e^{-x}}{2},$$
$$\sinh x = \frac{e^x - e^{-x}}{2},$$

respectively.[1] It is straightforward to check that they satisfy the identity

$$\cosh^2 x - \sinh^2 x = 1$$

as well as the derivative formulae

$$\frac{d}{dx}\cosh x = \sinh x,$$
$$\frac{d}{dx}\sinh x = \cosh x.$$

The names for these functions arise from the fact that they parametrize the (right branch) H of the hyperbola $x^2 - y^2 = 1$ in the same manner that the circular functions sine and cosine parametrize the circle $x^2 + y^2 = 1$. Namely, if we draw a ray R from the origin into either the first or fourth quadrants, and let A denote the area trapped by R, H and the x-axis (A is taken to be negative if R has negative slope), as shown in the diagram below, then the coordinates of the intersection of R and H can be shown to be

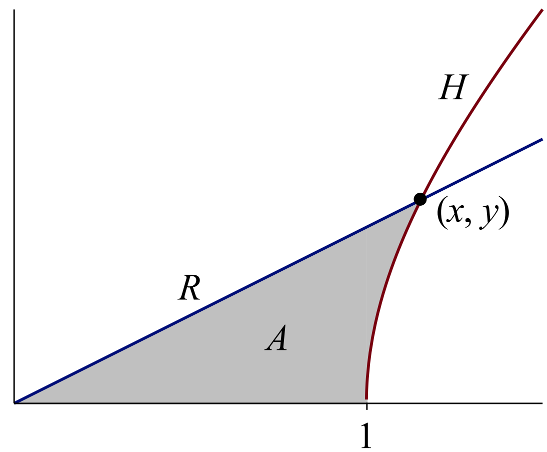
$$x = \cosh 2A,$$
$$y = \sinh 2A.$$

If we perform the same construction using the unit circle C instead, which is given by $x^2 + y^2 = 1$, we obtain

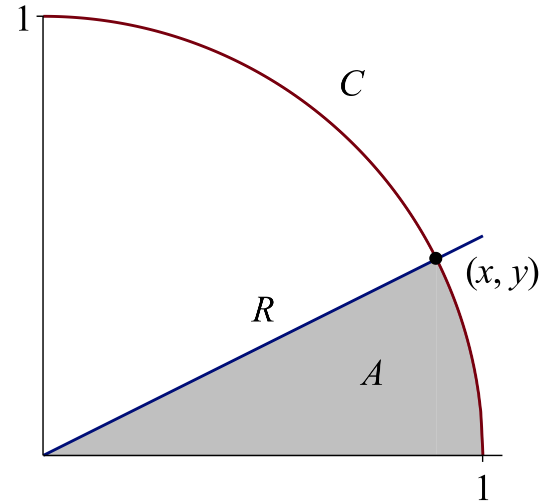
$$x = \cos 2A,$$
$$y = \sin 2A.$$

So the analogy between the circular and hyperbolic functions is that they parametrize different curves, but in the exact same way.

[1] These are typically read as "kosh of x" and "cinch of x."



The parametrization of the hyperbola H by the area A, resulting in the hyperbolic functions.



The parametrization of the unit circle C by the area A, resulting in the circular functions.

2 Application: Solutions of Second Order Homogeneous Linear Constant Coefficient ODEs

Consider the homogeneous linear second order ODE

$$ay'' + by' + cy = 0. \tag{1}$$

Suppose that the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

has two distinct real roots. According to the quadratic formula, these are given by

$$\frac{-b \pm \sqrt{\Delta}}{2a}$$

where $\Delta = b^2 - 4ac > 0$ is the *discriminant* of (2).

By the general theory of the solutions to equations of the form (1), the functions

$$y_1 = \exp\left(\frac{-b + \sqrt{\Delta}}{2a}x\right) \quad \text{and} \quad y_2 = \exp\left(\frac{-b - \sqrt{\Delta}}{2a}x\right)$$

form a basis for the solution space. In particular,

$$\widehat{y_1} = \frac{y_1 + y_2}{2} = e^{-bx/2a}\cosh\left(\frac{\sqrt{\Delta}}{2a}x\right) \quad \text{and} \quad \widehat{y_2} = \frac{y_1 - y_2}{2} = e^{-bx/2a}\sinh\left(\frac{\sqrt{\Delta}}{2a}x\right)$$

are both solutions of (1). We contend that $\widehat{y_1}$ and $\widehat{y_2}$ also form a basis for the space of solutions to (1). Indeed, we have the relationship

$$\begin{pmatrix} \widehat{y_1} \\ \widehat{y_2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and since the determinant of the matrix is $-1/2 \neq 0$ we can invert it to express any linear combination of y_1 and y_2 as a linear combination of $\widehat{y_1}$ and $\widehat{y_2}$, and vice versa. We have therefore proven the following useful result.

Theorem 1. *If the characteristic equation* (2) *has distinct real roots*

$$\frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac > 0,$$

then the general solution to (1) *is*

$$y = e^{-bx/2a}\left(c_1 \cosh\left(\frac{\sqrt{\Delta}}{2a}x\right) + c_2 \sinh\left(\frac{\sqrt{\Delta}}{2a}x\right)\right),$$

and every pair (c_1, c_2) *yields a distinct solution.*

Example 1. Find the general solution to $y'' - 4y' + y = 0$.

The characteristic equation is

$$r^2 - 4r + 1 = 0 \quad \Rightarrow \quad r = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

So Theorem 1 tells us the general solution is given by

$$\boxed{y = e^{2x}\left(c_1 \cosh\left(\sqrt{3}\,x\right) + c_2 \sinh\left(\sqrt{3}\,x\right)\right).}$$

♦

If Δ is a perfect square then often times the roots of (2) can be found by factoring the polynomial, and it might be difficult to identify $-b/2a$ and $\sqrt{\Delta}/2a$ in order to express the solutions of (1) as in Theorem 1. But we can find them as follows. Suppose the roots of (2) are $r_1 > r_2$. Then

$$ar^2 + br + c = a(r - r_1)(r - r_2) = a(r^2 - (r_1 + r_2)r + r_1 r_2).$$

Comparing the coefficients on either end of this equation we find that

$$b = -a(r_1 + r_2) \quad \text{and} \quad c = ar_1 r_2.$$

We immediately find that $-b/2a = (r_1 + r_2)/2$, the average of the roots. Moreover

$$\Delta = b^2 - 4ac = a^2(r_1^2 + 2r_1 r_2 + r_2^2) - 4a^2 r_1 r_2 = a^2(r_1 - r_2)^2$$

so that $\sqrt{\Delta} = |a|(r_1 - r_2)$ and

$$\frac{\sqrt{\Delta}}{2a} = \frac{|a| \cdot (r_1 - r_2)}{2a} = \sigma(a)\frac{r_1 - r_2}{2} = \sigma(a)\frac{r_1 - r_2}{2},$$

where $\sigma(a) \in \{\pm 1\}$ is the *sign* of a. However, the hyperbolic cosine and sine are even and odd, respectively, so that we may either ignore the sign or factor it out. But in the latter case the sign can simply be absorbed into the constant c_2. We can therefore recast Theorem 1 as follows.

Theorem 2. *If the characteristic equation of* (1) *has distinct real roots* $r_1 > r_2$*, then the general solution to* (1) *is given by*

$$y = e^{(r_1+r_2)x/2}\left(c_1 \cosh\left(\frac{r_1 - r_2}{2}x\right) + c_2 \sinh\left(\frac{r_1 - r_2}{2}x\right)\right),$$

and every pair (c_1, c_2) *yields a distinct solution.*

Example 2. Solve $9y'' + 3y' - 2y = 0$.

The polynomial occurring in the characteristic equation factors easily:

$$9r^2 + 3r - 2 = (3r + 2)(3r - 1)$$

so the solutions to the characteristic equation are $1/3$ and $-2/3$. We have $r_1 + r_2 = -1/3$ and $r_1 - r_2 = 1$. Hence the general solution is

$$\boxed{y = e^{-x/6}\left(c_1 \cosh\left(\frac{x}{2}\right) + c_2 \sinh\left(\frac{x}{2}\right)\right).}$$

♦

3 The Connection Between the Hyperbolic and Circular Functions

Recall *Euler's formula:*

$$e^{ix} = \cos x + i \sin x. \tag{3}$$

If we replace x with $-x$ and use the parity of sine and cosine we obtain

$$e^{-ix} = \cos x - i \sin x. \tag{4}$$

If we add (3) to (4) and divide by 2 we arrive at the fundamental relationship

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \cosh ix. \tag{5}$$

Subtracting (4) from (3) and this time dividing by $2i$ yields

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = -i \sinh ix. \tag{6}$$

If we substitute $-ix$ for x in these two results we find the complementary relationships

$$\cosh x = \cos ix,$$
$$\sinh x = -i \sin ix.$$

Let $z = x + iy$ be an arbitrary complex number with $x, y \in \mathbb{R}$. We define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

and through this definition we set

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

extending (5) and (6) to complex arguments.

Notice that

$$\begin{aligned}
\cos z = \frac{e^{iz} + e^{-iz}}{2} &= \frac{e^{ix-y} + e^{-ix+y}}{2} \\
&= \frac{e^{ix-y} + e^{ix+y} - e^{ix+y} + e^{-ix+y}}{2} \\
&= e^{ix} \cosh y - ie^{y} \sin x \\
&= \cos x \cosh y + i \sin x \left(\cosh y - e^{y}\right) \\
&= \cos x \cosh y - i \sin x \sinh y,
\end{aligned}$$

which gives the real and imaginary parts of $\cos z$ in terms of the real and imaginary parts of z. In a similar fashion one can establish that

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

4 ODEs with Negative Discriminant

Suppose now that $\Delta = b^2 - 4ac < 0$ in (1) so that the roots of (2) are complex conjugates. Let's proceed formally and apply the conclusion of Theorem 1 to (1). Using (5) and (6) we end up with the "solutions"

$$\begin{aligned}
y &= e^{-bx/2a} \left(c_1 \cosh\left(\frac{\sqrt{\Delta}}{2a} x\right) + c_2 \sinh\left(\frac{\sqrt{\Delta}}{2a} x\right)\right) \\
&= e^{-bx/2a} \left(c_1 \cosh\left(\frac{i\sqrt{|\Delta|}}{2a} x\right) + c_2 \sinh\left(\frac{i\sqrt{|\Delta|}}{2a} x\right)\right) \\
&= e^{-bx/2a} \left(c_1 \cos\left(\frac{\sqrt{|\Delta|}}{2a} x\right) + ic_2 \sin\left(\frac{\sqrt{|\Delta|}}{2a} x\right)\right),
\end{aligned}$$

which are only hypothetical since they are clearly complex-valued (if $c_2 \neq 0$). But if we absorb the imaginary unit i into c_2 and simply "forget" that this makes it non-real, we arrive at what we know are the genuine solutions to (1):

$$y = e^{-bx/2a} \left(c_1 \cos\left(\frac{\sqrt{|\Delta|}}{2a} x\right) + c_2 \sin\left(\frac{\sqrt{|\Delta|}}{2a} x\right)\right).$$

The Moral. Up to the fact that we have to allow c_2 to be imaginary when $\Delta < 0$, Theorem 1 applies in every situation in which the characteristic equation (2) has distinct roots.