Heat Conduction in a Rod with Inhomogeneous Neumann Boundary Conditions

R. C. Daileda

March 2, 2018

Consider the following heat problem,

$$\begin{aligned} &u_t = c^2 u_{xx}, && 0 < x < L,\ 0 < t, \\ &u_x(0,t) = -F_1, && 0 < t, \\ &u_x(L,t) = -F_2, && 0 < t, \\ &u(x,0) = f(x), && 0 < x < L, \end{aligned}$$

in which we are enforcing (potentially) inhomogeneous Neumann boundary conditions at the ends of the rod. If $\phi(x,t)$ denotes the (rightward) *heat flux* (thermal energy per unit time flowing to the right) throughout the rod (at position x and time t), then *Fourier's law of heat conduction* states that

$$\phi(x,t) = -K_0 u_x(x,t), \tag{1}$$

where $K_0 > 0$ is the *thermal conductivity* of the rod material. Fourier's law is a quantitative formulation of the fact that thermal energy moves "downhill," i.e. from hotter regions to cooler regions.

In light of Fourier's law, we find that our boundary conditions are equivalent to

$$\begin{aligned} \phi(0,t) &= K_0 F_1, \\ \phi(L,t) &= K_0 F_2, \end{aligned}$$

for all $t > 0$. That is, the amount of energy per unit time flowing *into* the left end of the rod is constantly $K_0 F_1$ while the amount per unit time flowing *out of* the right end is constantly $K_0 F_2$.[1] Since the rates at which thermal energy enters and leaves the ends of rod do not depend on time, we suspect that the same will (eventually) hold *throughout* the rod, that is

$$\phi(x,t) = \phi(x).$$

Substituting a time-independent heat flux into Fourier's law (1) and integrating with respect to x tell us that

$$u(x,t) = f(x) + g(t),$$

at least eventually. Substitution of this expression for the temperature into the heat equation yields

$$g'(t) = c^2 f''(x) \quad \Rightarrow \quad \text{both sides are constant.}$$

[1] If F_1 or F_2 is negative, so is the heat flux, which means thermal energy is actually moving to the *left*.

Setting $f''(x) = A$ we find that $g'(t) = c^2 A$, and integration of these expressions gives us

$$f(x) = \frac{A}{2}x^2 + Bx + C,$$
$$g(t) = c^2 At + D,$$

for some constants B, C and D. Since $u = f + g$, imposing the initial condition at $x = 0$ and $x = L$ tells us that we need

$$-F_1 = u_x(0,t) = f'(0) = B,$$
$$-F_2 = u_x(L,t) = f'(L) = AL + B.$$

We immediately find that $A = \frac{F_1 - F_2}{L}$ and hence that

$$f(x) = \frac{F_1 - F_2}{2L}x^2 - F_1 x + C.$$

Therefore

$$u(x,t) = f(x) + g(t) = \frac{F_1 - F_2}{2L}x^2 - F_1 x + \frac{c^2(F_1 - F_2)}{L}t + \underbrace{(C + D)}_{C_0}. \qquad (2)$$

This represents the eventual state of the temperature throughout the rod, although we cannot determine C_0 without additional information.

If we subtract C_0 from the solution (2), superposition guarantees that u will still solve the heat equation and satisfy the constant heat flux boundary conditions. This yields what we will call the *homogenizer* of the original boundary value problem:

$$u_1(x,t) = \frac{F_1 - F_2}{2L}x^2 - F_1 x + \frac{c^2(F_1 - F_2)}{L}t. \qquad (3)$$

We have reasoned heuristically that, up to a constant, this gives the long term evolution of the temperature in the rod, a fact that we will soon derive analytically.

If we let $v = u - u_1$, where u is the solution to the original problem, then superposition tells us that v satisfies

$$\begin{array}{ll} v_t = c^2 v_{xx}, & 0 < x < L,\ 0 < t, \\ v_x(0,t) = 0, & 0 < t, \\ v_x(L,t) = 0, & 0 < t, \\ v(x,0) = f(x) - u_1(x,0), & 0 < x < L, \end{array}$$

a heat conduction problem with *homogeneous* Neumann boundary conditions and a modified initial condition. Hence the name for u_1.

We know (see Example 1 of section 3.6 of our text) that if

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

is the $2L$-periodic cosine expansion of the initial condition $f(x) - u_1(x,0)$, then v is given by

$$v(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi x}{L}\right),$$

where $\lambda_n = cn\pi/L$. It finally follows that

$$u(x,t) = u_1(x,t) + v(x,t) = \frac{F_1 - F_2}{2L}x^2 - F_1 x + \frac{c^2(F_1 - F_2)}{L}t + a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi x}{L}\right).$$

Because of the decaying exponential factors, one can show that the series tends (uniformly) to 0 as $t \to \infty$, which means that the long term behavior of the temperature is given by

$$u_\infty(x,t) = \frac{F_1 - F_2}{2L}x^2 - F_1 x + \frac{c^2(F_1 - F_2)}{L}t + a_0, \tag{4}$$

as we expected.

Note that we have finally determined $C + D$ from (2): it's just

$$a_0 = \frac{1}{L}\int_0^L f(x) - u_1(x,0)\,dx = \widehat{a_0} + \frac{L(2F_1 + F_2)}{6}.$$

Here $\widehat{a_0}$ is the constant coefficient in the $2L$-periodic cosine expansion of the initial temperature profile $f(x)$, which is just the average initial temperature in the rod. By using the expression (3) for the homogenizer and the integral formulae for cosine expansion coefficients one can likewise show that

$$a_n = \widehat{a_n} + \frac{2L\left((-1)^n F_2 - F_1\right)}{\pi^2 n^2}$$

with $\widehat{a_n}$ denoting the nth coefficient of the $2L$-periodic cosine expansion of $f(x)$. Consequently we once again see the phenomenon of *a priori* knowledge of the half-range cosine expansion of $f(x)$ yielding the solution to the original boundary value problem almost immediately.

The behavior of u_∞ is relatively simple and easy to understand. Equation (4) shows that:

1. If $F_1 > F_2$, then the graph of $u_\infty(x,t)$ versus x is an upward opening parabola that moves upward with a speed of $c^2(F_1 - F_2)/L$. This occurs because the rate at which thermal energy enters the left end of the rod exceeds the rate at which it leaves the right end.

2. If $F_1 < F_2$, then the graph of $u_\infty(x,t)$ versus x is a downward opening parabola that moves downward ward with a speed of $c^2(F_1 - F_2)/L$. This occurs because the rate at which thermal energy leaves the right end of the rod exceeds the rate at which it enters the left end.

3. If $F_1 = F_2$, then u_∞ is the steady state

$$-F_1 x + \widehat{a_0} + \frac{LF_1}{2},$$

which is a line of slope $-F_1$ that passes through the average initial temperature at the midpoint of the rod. We get an eventual steady state because thermal energy is entering and leaving the two ends of the rod at exactly the same rate.