The Laplacian in Polar Coordinates

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Partial Differential Equations
Lecture 13
To solve boundary value problems on circular regions, it is convenient to switch from rectangular \((x, y)\) to polar \((r, \theta)\) spatial coordinates:

\[
x = r \cos \theta, \\
y = r \sin \theta, \\
x^2 + y^2 = r^2.
\]

This requires us to express the rectangular Laplacian

\[
\Delta u = u_{xx} + u_{yy}
\]

in terms of derivatives with respect to \(r\) and \(\theta\).
The chain rule

For any function $f(r, \theta)$, we have the familiar tree diagram and chain rule formulae:

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x}
\]

\[
\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial y}
\]

or

\[
f_x = f_r r_x + f_\theta \theta_x
\]

\[
f_y = f_r r_y + f_\theta \theta_y
\]
First take \( f = u \) to obtain

\[
u_x = u_r r_x + u_\theta \theta_x \quad \Rightarrow \quad u_{xx} = u_r r_{xx} + (u_r)_x r_x + u_\theta \theta_{xx} + (u_\theta)_x \theta_x.
\]

Applying the chain rule with \( f = u_r \) and then with \( f = u_\theta \) yields

\[
\begin{align*}
u_{xx} &= u_r r_{xx} + (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_\theta \theta_{xx} + (u_{\theta r} r_x + u_{\theta\theta} \theta_x) \theta_x \\
&= u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta} \theta_x^2.
\end{align*}
\]

An entirely similar computation using \( y \) instead of \( x \) also gives

\[
\begin{align*}
u_{yy} &= u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta} \theta_y^2.
\end{align*}
\]

If we add these expressions and collect like terms we get

\[
\Delta u = u_r (r_{xx} + r_{yy}) + u_{rr} (r_x^2 + r_y^2) + 2u_{r\theta} (r_x \theta_x + r_y \theta_y) \\
+ u_\theta (\theta_{xx} + \theta_{yy}) + u_{\theta\theta} (\theta_x^2 + \theta_y^2).
\]
Differentiate $x^2 + y^2 = r^2$ with respect to $x$:

$$2x = 2rr_x \Rightarrow r_x = \frac{x}{r} \Rightarrow r_{xx} = \frac{r - xr_x}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3},$$

and by symmetry

$$r_y = \frac{y}{r} \text{ and } r_{yy} = \frac{x^2}{r^3}.$$

Now differentiate $\tan \theta = \frac{y}{x}$ with respect to $x$ and then $y$:

$$\sec^2 \theta \theta_x = -\frac{y}{x^2} \Rightarrow \theta_x = -\frac{y \cos^2 \theta}{x^2} = -\frac{y}{r^2} \Rightarrow \theta_{xx} = \frac{2y}{r^3}r_x = \frac{2xy}{r^4},$$

$$\sec^2 \theta \theta_y = \frac{1}{x} \Rightarrow \theta_y = \frac{\cos^2 \theta}{x} = \frac{x}{r^2} \Rightarrow \theta_{yy} = \frac{-2x}{r^3}r_y = -\frac{2xy}{r^4}.$$
Together these yield

\[ r_{xx} + r_{yy} = \frac{y^2 + x^2}{r^3} = \frac{1}{r}, \quad r_x^2 + r_y^2 = \frac{x^2 + y^2}{r^2} = 1. \]

\[ \theta_{xx} + \theta_{yy} = \frac{2xy}{r^4} + \frac{-2xy}{r^4} = 0, \quad \theta_x^2 + \theta_y^2 = \frac{y^2 + x^2}{r^4} = \frac{1}{r^2}, \]

\[ r_x \theta_x + r_y \theta_y = \frac{-xy}{r^3} + \frac{yx}{r^3} = 0, \]

and we finally obtain

\[ \Delta u = u_r (r_{xx} + r_{yy}) + u_{rr} (r_x^2 + r_y^2) + 2u_r \theta (r_x \theta_x + r_y \theta_y) \]
\[ + u_\theta (\theta_{xx} + \theta_{yy}) + u_{\theta \theta} (\theta_x^2 + \theta_y^2) \]
\[ = \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = u_{rr} + \frac{1}{r^2} u_{\theta \theta}. \]
Example

Use polar coordinates to show that the function \( u(x, y) = \frac{y}{x^2 + y^2} \) is harmonic.

We need to show that \( \Delta u = 0 \). In polar coordinates we have

\[
u(r, \theta) = \frac{r \sin \theta}{r^2} = \frac{\sin \theta}{r}\]

so that

\[
u_r = -\frac{\sin \theta}{r^2}, \quad u_{rr} = \frac{2 \sin \theta}{r^3}, \quad u_{\theta \theta} = \frac{-\sin \theta}{r},\]

and thus

\[
\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \frac{2 \sin \theta}{r^3} - \frac{\sin \theta}{r^3} - \frac{\sin \theta}{r^3} = 0.
\]
The Dirichlet problem on a disk

**Goal:** Solve the Dirichlet problem on a disk of radius \(a\), centered at the origin. In polar coordinates this has the form

\[
\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 \leq r < a,
\]

\[
u(a, \theta) = f(\theta), \quad 0 \leq \theta \leq 2\pi.
\]

**Remarks:**

- We will require that \(f\) is \(2\pi\)-periodic.
- Likewise, we require that \(u(r, \theta)\) is \(2\pi\)-periodic in \(\theta\).
Separation of variables

If we assume that $u(r, \theta) = R(r)\Theta(\theta)$ and plug into $\Delta u = 0$, we get

$$R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0 \quad \Rightarrow \quad r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

Thus, we have

$$\Rightarrow \quad r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

This yields the pair of separated ODEs

$$r^2 R'' + r R' - \lambda R = 0 \quad \text{and} \quad \Theta'' + \lambda \Theta = 0.$$

We also have the “boundary conditions”

$$\Theta \text{ is } 2\pi\text{-periodic} \quad \text{and} \quad R(0^+) \text{ is finite.}$$
Solving for $\Theta$

The solutions of $\Theta'' + \lambda \Theta = 0$ are periodic only if

$$\lambda = \mu^2 \geq 0 \implies \Theta = a \cos(\mu \theta) + b \sin(\mu \theta).$$

In order for the period to be $2\pi$ we also need

$$1 = \cos(0\mu) = \cos(2\pi\mu) \implies 2\pi \mu = 2\pi n \implies \mu = n \in \mathbb{N}_0.$$

Hence $\lambda = n^2$ and

$$\Theta = \Theta_n = a_n \cos(n \theta) + b_n \sin(n \theta), \quad n \in \mathbb{N}_0.$$

It follows that $R$ satisfies

$$r^2 R'' + r R' - n^2 R = 0,$$

which is called an *Euler equation*. 
Interlude

Euler equations

An *Euler equation* is a second order ODE of the form

\[ x^2 y'' + \alpha xy' + \beta y = 0. \]

Its solutions are determined by the roots of its *indicial equation*

\[ \rho^2 + (\alpha - 1)\rho + \beta = 0. \]

**Case 1:** If the roots are \( \rho_1 \neq \rho_2 \), then the general solution is

\[ y = c_1 x^{\rho_1} + c_2 x^{\rho_2}. \]

**Case 2:** If there is only one root \( \rho_1 \), then the general solution is

\[ y = c_1 x^{\rho_1} + c_2 x^{\rho_1} \ln x. \]
Changing to polar coordinates

The Dirichlet problem on a disk

Examples

Solving for $R$

The indicial equation of $r^2 R'' + rR' - n^2 R = 0$ is

$$\rho^2 + (1 - 1)\rho - n^2 = \rho^2 - n^2 = 0 \iff \rho = \pm n.$$ 

This means that

$$R = c_1 r^n + c_2 r^{-n} \quad (n \neq 0),$$

$$R = c_1 + c_2 \ln r \quad (n = 0).$$

These will be finite at $r = 0$ only if $c_2 = 0$. Setting $c_1 = a^{-n}$ gives

$$R = R_n = \left(\frac{r}{a}\right)^n, \quad n \in \mathbb{N}_0.$$
Changing to polar coordinates

We therefore obtain the separated solutions

\[ u_n(r, \theta) = R_n(r)\Theta_n(\theta) = \left(\frac{r}{a}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)), \quad n \in \mathbb{N}_0. \]

Noting that

\[ u_0(r, \theta) = \left(\frac{r}{a}\right)^0 (a_0 \cos 0 + b_0 \sin 0) = a_0, \]

superposition gives the general solution

\[ u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)). \]
Boundary values and conclusion

Imposing our Dirichlet boundary conditions gives

\[ f(\theta) = u(a, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)), \]

which is just the ordinary 2\(\pi\)-periodic Fourier series for \(f\)!

**Theorem**

The solution of the Dirichlet problem on the disk of radius \(a\) centered at the origin, with boundary condition \(u(a, \theta) = f(\theta)\) is

\[ u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)), \]

where

\[ a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) \, d\theta, \]

\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \cos(n\theta) \, d\theta, \quad b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \sin(n\theta) \, d\theta. \]
Example

Find the solution to the Dirichlet problem on a disk of radius 3 with boundary values given by

\[
f(\theta) = \begin{cases} 
\frac{30}{\pi} (\pi + 2\theta) & \text{if } -\frac{\pi}{2} \leq \theta < 0, \\
\frac{30}{\pi} (\pi - 2\theta) & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\
0 & \text{if } \frac{\pi}{2} \leq \theta < \frac{3\pi}{2}.
\end{cases}
\]

We have \( a = 3 \). The graph of \( f \) is
According to exercise 2.3.8 (with \( p = \pi \), \( c = 30 \) and \( d = \pi/2 \)):

\[
f(\theta) = \frac{15}{2} + \frac{120}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi/2)}{n^2} \cos(n\theta).
\]

Hence, the solution to the Dirichlet problem is

\[
u(r, \theta) = \frac{15}{2} + \frac{120}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{r}{3} \right)^n \frac{1 - \cos(n\pi/2)}{n^2} \cos(n\theta).
\]
Example

Solve the Dirichlet problem on a disk of radius 2 with boundary values given by \( f(\theta) = \cos^2 \theta \). Express your answer in cartesian coordinates.

We have \( a = 2 \) and

\[
f(\theta) = \cos^2 \theta = \frac{1 + \cos(2\theta)}{2} = \frac{1}{2} + \frac{1}{2} \cos(2\theta),
\]

which is a finite \( 2\pi \)-periodic Fourier series (i.e. \( a_0 = 1/2 \), \( a_2 = 1/2 \), and all other coefficients are zero).

Hence

\[
u(r, \theta) = \frac{1}{2} + \left(\frac{r}{2}\right)^2 \cdot \frac{1}{2} \cos(2\theta) = \frac{1}{2} + \frac{r^2 \cos(2\theta)}{8}.
\]
Since \( \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \), we find that

\[
r^2 \cos(2\theta) = r^2 \cos^2 \theta - r^2 \sin^2 \theta = x^2 - y^2
\]

and hence

\[
u = \frac{1}{2} + \frac{r^2 \cos(2\theta)}{8} = \frac{1}{2} + \frac{x^2 - y^2}{8}.
\]
Example

Solve the Dirichlet problem on a disk of radius 1 if the boundary value is 50 in the first quadrant, and zero elsewhere.

We are given \( a = 1, \ f(\theta) = 50 \) for \( 0 \leq \theta \leq \pi/2 \) and \( f(\theta) = 0 \) otherwise. The Fourier coefficients of \( f \) are

\[
a_0 = \frac{1}{2\pi} \int_0^{\pi/2} 50 \, d\theta = \frac{25}{2},
\]

\[
a_n = \frac{1}{\pi} \int_0^{\pi/2} 50 \cos(n\theta) \, d\theta = \frac{50 \sin(n\pi/2)}{n\pi},
\]

\[
b_n = \frac{1}{\pi} \int_0^{\pi/2} 50 \sin(n\theta) \, d\theta = \frac{50(1 - \cos(n\pi/2))}{n\pi},
\]

so that

\[
u(r, \theta) = \frac{25}{2} + \frac{50}{\pi} \sum_{n=1}^{\infty} r^n \left( \frac{\sin(n\pi/2)}{n} \cos(n\theta) + \frac{1 - \cos(n\pi/2)}{n} \sin(n\theta) \right).
\]
Remarks:

- One can frequently use identities like (valid for $|r| < 1$)

$$\sum_{n=1}^{\infty} \frac{r^n \cos(n\theta)}{n} = -\frac{1}{2} \ln \left(1 - 2r \cos \theta + r^2\right),$$

$$\sum_{n=1}^{\infty} \frac{r^n \sin(n\theta)}{n} = \arctan \left(\frac{r \sin \theta}{1 - r \cos \theta}\right),$$

to convert series solutions in polar coordinates to cartesian expressions.

- Using the second identity, one can show that the solution in the preceding example is

$$u(x, y) = \frac{25}{2} + \frac{50}{\pi} \left(\arctan \left(\frac{y}{1 - x}\right) + \arctan \left(\frac{x}{1 - y}\right)\right).$$