Power Series, Part 1: Review of Basic Properties

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Partial Differential Equations
Lecture 14

The vibrating circular membrane

Goal: Model the motion of an elastic membrane stretched over a circular frame of radius a.

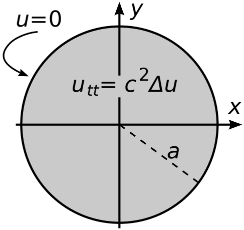
Set-up: Center the membrane at the origin in the xy-plane and let

$$u(r,\theta,t) = \begin{cases} \text{deflection of membrane from equilibrium at} \\ \text{polar position } (r,\theta) \text{ and time } t. \end{cases}$$

Under ideal assumptions:

$$u_{tt} = c^2 \Delta u = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right),$$
$$0 < r < a, \quad 0 < \theta < 2\pi, \quad t > 0,$$

$$u(a,\theta,t) = 0, \quad 0 \le \theta \le 2\pi, \quad t > 0.$$



Separation of variables

Setting $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ leads to the separated boundary value problems

$$r^2R'' + rR' + \left(\lambda^2r^2 - \mu^2\right)R = 0, \quad R(0+) \text{ finite}, \quad R(a) = 0,$$
$$\Theta'' + \mu^2\Theta = 0, \quad \Theta \ 2\pi\text{-periodic},$$
$$T'' + c^2\lambda^2T = 0.$$

We have already seen that the solutions to the Θ problem are

$$\Theta(\theta) = \Theta_m(\theta) = A\cos(m\theta) + B\sin(m\theta), \quad \mu = m \in \mathbb{N}_0.$$

So, for each $m \in \mathbb{N}_0$ it remains to solve the ODE boundary value problem

$$r^2R'' + rR' + \left(\lambda^2r^2 - m^2\right)R = 0, \quad R(0+) \text{ finite}, \quad R(a) = 0.$$

Solving for R

Case 1: $\lambda = 0$. This is an Euler equation, and the only solution to the BVP is $R \equiv 0$ (HW).

Case 2: $\lambda > 0$. The changes of variables $R(r) = y(x)$, $x = \lambda r$ lead to

$$\underbrace{x^2y'' + xy' + (x^2 - m^2)y = 0}_{\text{Bessel's equation of order } m}, \quad y(0+) \text{ finite}, \quad y(\lambda a) = 0.$$

Remarks.

- The solutions to Bessel's equation have been well-studied.
- However, in order to understand them we need to introduce a *new* technique for solving second order ODEs.
- We will spend the next several lectures studying the *Power Series* and *Frobenius Methods*.

Definition of a power series

A *power series [PS] (centered at a)* is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots,$$

where $a_0, a_1, a_2, a_3, \ldots$ are (real) constants called its *coefficients*.

Examples.

1. $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is a PS centered at $a = 0$.
2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ is a PS centered at $a = 0$.
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$ is a PS centered at $a = 1$.

Convergence of power series

For what x does a power series converge?

Theorem

Given a PS $f(x) = \sum_{n=1}^{\infty} a_n(x-a)^n$*, there is an* $R \in [0, \infty]$ *(its* radius of convergence*) so that*

- $f(x)$ *converges absolutely if* $|x-a| < R$ *and*
- $f(x)$ *diverges if* $|x-a| > R$*.*

The series may or may not *converge when* $|x-a| = R$*.*

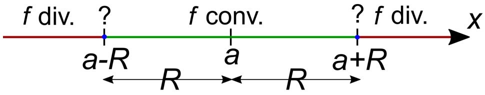
Corollary

Every PS centered at a converges on an interval (its interval of convergence*) centered at a of radius equal to R. A PS* may or may not *converge at the endpoints of its interval of convergence.*

Remarks

Consider the PS

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$



Using the ratio/root tests, one can show that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \text{OR} \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \Rightarrow \quad R = \frac{1}{L}.$$

Warnings:

- In some cases these limits *may not* exist, so they can't be used in every situation.
- Unless $L = 0, \infty$, these limits *never* provide information about the behavior of a PS at the endpoints.

Example

Find the interval of convergence of

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Using the root test alluded to above,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{1} = 1,$$

so that the radius is $R = 1/1 = 1$. Since the center is $a = 0$, we must test the endpoints $x = a \pm R = \pm 1$ directly.

When $x = \pm 1$, then nth term in the series is

$$(\pm 1)^n \not\to 0 \quad \text{as} \quad n \to \infty \quad \Rightarrow \quad \text{the series diverges.}$$

Hence the interval of convergence is $(-1, 1)$.

Example

Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{2^n n} = \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{24} - \cdots .$$

Taking the nth roots of the coefficients gives

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n+1}}{2^n n}\right|} = \lim_{n\to\infty} \frac{1}{2\sqrt[n]{n}} = \frac{1}{2}.$$

So the radius of convergence is $R = 1/(1/2) = 2$.

The series is centered at $a = 1$, so the endpoints $x = a \pm R = -1, 3$ must be checked directly.

When $x = -1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{2^n n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent (the negative of the harmonic series).

When $x = 3$ the series instead becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which converges by the alternating series test.

We conclude that the interval of convergence is $(-1, 3]$.

Example

Determine the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots .$$

Computing the ratios of the coefficients gives

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.\end{aligned}$$

So the radius of convergence is $R = 1/0 = \infty$, and the interval of convergence is $(-\infty, \infty) = \mathbb{R}$.

Remarks

- As with Fourier series, one can visualize the convergence of a power series by plotting

$$s_N(x) = \sum_{n=0}^{N} a_n(x-a)^n \qquad \text{(the } N\text{th partial sum)}$$

and letting $N \to \infty$.

- In fact, the partial sums converge *uniformly* to the PS on any closed subinterval of the interval of convergence that omits the endpoints.

Arithmetic of power series

Given two PS

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{with radius} \quad R_1 > 0,$$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n \quad \text{with radius} \quad R_2 > 0,$$

then their "formal" sum and product

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-a)^n,$$

$$(f \cdot g)(x) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0)(x-a)^n,$$

both converge with radii $R \geq \min\{R_1, R_2\}$.

Remarks

- Since constants can be regarded as power series with infinite radius (i.e. $\alpha = \alpha + 0(x-a) + 0(x-a)^2 + \cdots$), we can make analogous statements for linear combinations $\alpha f(x) + \beta g(x)$.
- If $g(a) \neq 0$ (i.e. $b_0 \neq 0$), one can also formally compute f/g as a power series using "polynomial" long division,and it will have a positive radius of convergence.
- A function equal to a power series centered at a with positive radius of convergence is called *analytic at a*.
- According to the previous slide, linear combinations, products and (appropriate) quotients of analytic functions are also analytic. Many familiar functions from Calculus are analytic (almost) everywhere.

First examples of analytic functions

- Recall from Calc. II that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1.$$

- According to the previous result

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for} \quad |x| < 1.$$

- Since $|x| < 1$ implies that $|x^2| < 1$ we also have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n (x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for} \quad |x| < 1.$$

Therefore $\frac{1}{1-x}$, $\frac{1}{1+x}$ and $\frac{1}{1+x^2}$ are all analytic at $a = 0$.

Calculus of power series

Being analytic makes a function extremely "nice." The following result quantifies this statement.

Theorem

Every power series converges to a differentiable (hence integrable) function inside *its interval of convergence. Moreover*

$$\left.\begin{array}{c} f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n \\ \text{has radius } R \end{array}\right\} \Rightarrow \begin{array}{c} f'(x)=\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}, \\ \int f(x)\,dx=\sum_{n=0}^{\infty}\frac{a_n(x-a)^{n+1}}{n+1}+C, \end{array}$$

and both have radius R as well.

Remarks

According to the previous result:

- A PS can be differentiated and integrated *term-by-term* without changing its *radius* of convergence.
- **Warning:** differentiation and integration of a PS may change convergence behavior at the *endpoints* of the interval of convergence.
- Derivatives and antiderivatives of an analytic function are themselves analytic. This yields the following result.

Corollary

If f is analytic at a, then f is infinitely differentiable in a neighborhood of a, and all of its derivatives are analytic at a as well, with a common radius of convergence.

Example

Show that for every $x \in \mathbb{R}$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hence e^x is analytic at $a = 0$.

We have already seen that $R = \infty$, so the PS converges for all x. Moreover, according to the theorem above, we have

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!},$$

where we have used the *index substitution* $m = n - 1$.

Since the index m is a "dummy" variable, we may replace it with n again. We have then shown that

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad y' = y \quad \Rightarrow \quad y = Ce^x$$

for some C. To solve for C we plug in $x = 0$:

$$1 + 0 + 0 + 0 + \cdots = y(0) = Ce^0 \quad \Rightarrow \quad C = 1.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for every $x \in \mathbb{R}$.

Example

Show that

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for $|x| < 1$*. Hence* $\arctan x$ *is analytic at* $a = 0$*.*

If $|x| < 1$, then we have seen that

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Integrating both sides then gives

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C \quad \text{for} \quad |x| < 1.$$

Setting $x = 0$ yields $C = 0$, and we're finished.

Example

Show that $\frac{1}{(1-x)^2}$ *is analytic at* $a = 0$ *by finding its power series representation centered there.*

For $|x| < 1$ we have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{m=0}^{\infty}(m+1)x^m,$$

where we have made the change of index $m = n - 1$ in the final equality.

This expresses $\frac{1}{(1-x)^2}$ as a PS centered at $a = 0$, proving it is analytic there.

Example

Show that $\frac{1}{1+x}$ is analytic at $a = 2$ by finding its power series representation centered there.

We have

$$\frac{1}{1+x} = \frac{1}{3+(x-2)} = \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{x-2}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x-2}{3}\right)^n,$$

provided $|-(x-2)/3| < 1 \Leftrightarrow |x-2| < 3$. The final sum is a power series:

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x-2}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{3^{n+1}}.$$

Since $\frac{1}{1+x}$ equals a power series centered at $a = 2$, it is analytic there.

As a preview of the Power Series Method, let's consider the following.

Example

Show that $y = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$ *solves the ODE* $y' + xy = 1$ *for* $x \in \mathbb{R}$.

That the given series has $R = \infty$ is left as HW. Consequently, for any x we have

$$y' + xy = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!(2n+1)}{(2n+1)!} x^{2n} + x \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n)!} x^{2n} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+2}}_{\text{sub. } n=m-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n)!} x^{2n} + \underbrace{\sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{m-1} (m-1)!}{(2m-1)!} x^{2m}}_{\text{replace ``dummy'' } m \text{ with } n}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n 2^n n!}{(2n)!} + \frac{(-1)^{n-1} 2^{n-1} (n-1)!}{(2n-1)!} \right) x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n 2^n n(n-1)!}{(2n)(2n-1)!} + \frac{(-1)^{n-1} 2^{n-1} (n-1)!}{(2n-1)!} \right) x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n 2^{n-1} (n-1)!}{(2n-1)!} + \frac{(-1)^{n-1} 2^{n-1} (n-1)!}{(2n-1)!} \right) x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} 0 \cdot x^{2n} = 1,$$

which is what we needed to show.

Uniqueness of power series coefficients

By repeatedly differentiating and plugging in $x = a$, one can prove:

Theorem

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *has positive radius of convergence, then*

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \textit{for all} \quad n \geq 0.$$

This immediately yields:

Corollary (Identity Principle)

If $\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n$ *on an open interval containing* a, *then* $a_n = b_n$ *for all* $n \geq 0$.

Remarks

- This says that if f is analytic at a, there is *only one* power series (centered at a) that it can equal.
- This result also tells us that f is analytic at a if and only if

$$f(x) = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{Taylor series of } f \text{ at } a} \text{ for all } x \text{ near } a.$$

- Even f is known *a priori* to be analytic at a, one can frequently use algebraic manipulations of existing PS to avoid computing the Taylor series directly (as in earlier examples).