Power Series, Part 2:
Analytic Solutions at Ordinary Points of ODEs

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Partial Differential Equations
Lecture 15

Introductory example

Example

Find the general solution to $y'' - xy' - y = 0$.

This is a second order, linear ODE, but...

it does not have constant coefficients!

Consequently we *cannot* find the solutions in the usual way (via the characteristic polynomial).

Instead, we use the *Power Series Method*. We begin by *assuming* the solution is analytic at $a = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

and attempt to determine the coefficients a_n.

Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \;\; y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \;\; y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

plugging into the ODE we find that we must have

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{m=n-2} - \underbrace{x \sum_{n=1}^{\infty} n a_n x^{n-1}}_{\text{distribute } x} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m}_{n=m} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{2a_2 - a_0}_{x^0 \text{ coeff.}} + \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} - n a_n - a_n \right) x^n = 0.$$

Uniqueness of power series coefficients implies that

$$
\begin{aligned}
2a_2 - a_0 &= 0, \\
(n+2)(n+1)a_{n+2} - (n+1)a_n &= 0, \qquad n \geq 1,
\end{aligned}
$$

or, equivalently,

$$a_{n+2} = \frac{a_n}{n+2}, \quad n \geq 0.$$

Remarks.

- This is a *recursion relation* for the coefficients.
- We are free to choose a_0, a_1. Then $a_2, a_3, a_4, \ldots$ are completely determined.
- If possible, we would now like to solve for a_n in terms of a_0 and a_1.

Notice that

$$a_2 = \frac{a_0}{2} \quad \Rightarrow \quad a_4 = \frac{a_2}{4} = \frac{a_0}{4 \cdot 2} \quad \Rightarrow \quad a_6 = \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2} \quad \Rightarrow \quad \cdots$$

$$\Rightarrow \quad a_{2k} = \frac{a_0}{(2k)(2k-2)(2k-4)\cdots 2} = \frac{a_0}{2^k k!}$$

and

$$a_3 = \frac{a_1}{3} \quad \Rightarrow \quad a_5 = \frac{a_3}{5} = \frac{a_1}{5 \cdot 3} \quad \Rightarrow \quad a_7 = \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 3} \quad \Rightarrow \quad \cdots$$

$$\begin{aligned} \Rightarrow \quad a_{2k+1} &= \frac{a_1}{(2k+1)(2k-1)(2k-3)\cdots 3 \cdot 1} \\ &= \frac{(2k)(2k-2)(2k-4)\cdots 2a_1}{(2k+1)!} = \frac{2^k k! a_1}{(2k+1)!}. \end{aligned}$$

This means that

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=1}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= \underbrace{a_0 \sum_{k=1}^{\infty} \frac{1}{2^k k!} x^{2k}}_{y_1(x)} + \underbrace{a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}}_{y_2(x)}.$$

Remarks

- The ratio test implies both series have $R = \infty$ (HW).
- We have

$$\left.\begin{array}{lll} a_0 = 1, a_1 = 0 & \Rightarrow & y = y_1 \\ a_0 = 0, a_1 = 1 & \Rightarrow & y = y_2 \end{array}\right\} \Rightarrow \ y_1, y_2 \text{ both solve the ODE.}$$

Finally, notice that

$$\left.\begin{array}{ll} y_1(0)=1, & y_1'(0)=0 \\ y_2(0)=0, & y_2'(0)=1 \end{array}\right\} \Rightarrow \underbrace{W(y_1,y_2)}_{\text{the Wronskian}}(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

so that y_1 and y_2 are *linearly independent* solutions of the ODE.

Conclusion: The general solution to $y'' - xy' - y = 0$ is

$$y = a_0 y_1 + a_1 y_2 = a_0 \sum_{k=1}^{\infty} \frac{1}{2^k k!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1},$$

and these series converge for all x.

The following result generalizes the outcome of the previous example.

Theorem

Suppose that $p(x)$, $q(x)$ and $g(x)$ are analytic at $x = a$ and have (positive) radii of convergence R_1, R_2 and R_3, respectively. Then every solution of

$$y'' + p(x)y' + q(x)y = g(x)$$

is analytic at $x = a$ with radius $R \geq \min\{R_1, R_2, R_3\}$.

For the equation $y'' - xy' - y = 0$ we have

$$p(x) = -x, \;\; q(x) = -1, \;\; g(x) = 0.$$

These are all analytic at $a = 0$ with $R = \infty$, so the solutions must have the same property.

Remarks

- If $p(x)$, $q(x)$, and $g(x)$ are analytic at $x = a$, we say that $x = a$ is an *ordinary point* of $y'' + p(x)y' + q(x)y = g(x)$.
- Recall that if $y = \sum_{n=0}^{\infty} a_n(x-a)^n$, then

  $$a_0 = y(a) \quad \text{and} \quad a_1 = y'(a).$$

  Therefore the pair (a_0, a_1) will *always* determine the remaining coefficients of y. The Method of Power series provides an explicit recursion for the coefficients.
- If y_1 has $a_0 = 1, a_1 = 0$ and y_2 has $a_0 = 0, a_1 = 1$, then

  $$W(y_1, y_2)(a) = \begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

  $$\Rightarrow \quad y_1, y_2 \text{ are linearly independent solutions.}$$

Remarks (cont.)

With the preceding choices of y_1 and y_2:

- The general solution is $y = c_1 y_1 + c_2 y_2$.
- This general solution has the property that

$$y(a) = c_1 y_1(a) + c_2 y_2(a) = c_1,$$
$$y'(a) = c_1 y_1'(a) + c_2 y_2'(a) = c_2.$$

- This makes solving the IVP $y(a) = A$, $y'(a) = B$ extremely easy. If you've found y_1, y_2, then set

$$y = Ay_1 + By_2.$$

Otherwise, set $a_0 = A$, $a_1 = B$ and use the recursion relation.

Example

Show that $a = 0$ is an ordinary point of $(4 - x^2)y'' + 2y = 0$. Find two linearly independent solutions that are analytic at $a = 0$, and give a lower bound for their radii of convergence.

In standard form, the ODE is

$$y'' + \frac{2}{4 - x^2} y = 0.$$

We have $p(x) = 0$ and

$$q(x) = \frac{2}{4 - x^2} = \frac{1}{2} \cdot \frac{1}{1 - (x/2)^2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}} \quad \text{for } |x| < 2.$$

It follows that $a = 0$ is an ordinary point and that all solutions are analytic there with $R \geq 2$.

To find the solutions, we set $y = \sum_{n=0}^{\infty} a_n x^n$ in the ODE:

$$(4 - x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0,$$

$$\sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0,$$

$$\underbrace{8a_2 + 2a_0}_{n=0} + \underbrace{(24a_3 + 2a_1)x}_{n=1}$$

$$+ \sum_{n=2}^{\infty} \left(4(n+2)(n+1) a_{n+2} - (n(n-1) - 2) a_n\right) x^n = 0$$

This gives the relations

$$8a_2 + 2a_0 = 0, \;\; 24a_3 + 2a_1 = 0,$$
$$4(n+2)(n+1)a_{n+2} - \underbrace{\left(n(n-1) - 2\right)}_{(n-2)(n+1)} a_n = 0 \quad \text{for } n \geq 2,$$

or, equivalently

$$a_{n+2} = \frac{(n-2)a_n}{4(n+2)} \quad \text{for } n \geq 0.$$

Thus

$$a_2 = \frac{-a_0}{4} \;\Rightarrow\; a_4 = \frac{0 \cdot a_2}{4 \cdot 4} = 0 \;\Rightarrow\; a_6 = \frac{2 \cdot a_4}{4 \cdot 6} = 0 \;\Rightarrow\; \cdots$$

$$\Rightarrow a_{2k} = 0 \quad \text{for } k \geq 2.$$

And

$$a_3 = \frac{-a_1}{4 \cdot 3} \Rightarrow a_5 = \frac{a_3}{4 \cdot 5} = \frac{-a_1}{4^2 \cdot 5 \cdot 3} \Rightarrow a_7 = \frac{3 \cdot a_5}{4 \cdot 7} = \frac{-a_1}{4^3 \cdot 7 \cdot 5}$$
$$\Rightarrow a_9 = \frac{5 \cdot a_7}{4 \cdot 9} = \frac{-a_1}{4^4 \cdot 9 \cdot 7} \Rightarrow \cdots \Rightarrow a_{2k+1} = \frac{-a_1}{4^k(2k+1)(2k-1)}$$

for $k \geq 0$. Therefore, setting $a_0 = 1$, $a_1 = 0$ gives the solution

$$y_1(x) = 1 - \frac{x^2}{4} \quad (\text{note that } R = \infty)$$

and setting $a_0 = 0$, $a_1 = 1$ gives the independent solution

$$y_2(x) = -\sum_{k=0}^{\infty} \frac{x^{2k+1}}{4^k(2k+1)(2k-1)} \quad (\text{can show that } R = 2).$$

The general solution is $y = a_0 y_1 + a_1 y_2$.

Example

Show that $a = 0$ is an ordinary point of $y'' - 2y' + xy = 0$, and find the recursion relation satisfied by the coefficients of any solution that is analytic at $a = 0$. Determine the first few coefficients in two linearly independent solutions, and state their radii of convergence.

We have $p(x) = -2$ and $q(x) = x$, both of which are power series (at $a = 0$) with $R = \infty$.

Therefore $a = 0$ is an ordinary point, and every solution is analytic at $a = 0$ with $R = \infty$ as well.

Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ and simplifying yields

$$\underbrace{2a_2 - 2a_1}_{n=0} + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_{n-1}\right) x^n = 0.$$

Therefore $a_2 = a_1$ and

$$a_{n+2} = \frac{2(n+1)a_{n+1} - a_{n-1}}{(n+2)(n+1)} \quad \text{for } n \geq 1.$$

With $a_0 = 1$, $a_1 = 0$ this yields

$$y_1(x) = 1 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{30}x^5 - \frac{1}{80}x^6 + \frac{1}{2520}x^7 + \cdots,$$

whereas with $a_0 = 0$, $a_1 = 1$ we get

$$y_2(x) = x + x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 - \frac{1}{180}x^6 - \frac{19}{2520}x^7 - \cdots.$$

Example

Show that $a = 1$ is an ordinary point of $xy'' + y' + xy = 0$, and find the recursion relation satisfied by the coefficients of any solution that is analytic at $a = 1$. Determine the first few coefficients in two linearly independent solutions, and give a lower bound on their radii of convergence. Express the solution with initial conditions $y(1) = 5$, $y'(1) = -3$ in terms of this basis.

In standard form, the ODE is $y'' + \frac{1}{x}y' + y = 0$, which has $q(x) = 1$ and

$$p(x) = \frac{1}{x} = \frac{1}{1 + (x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \text{for } |x-1| < 1.$$

Therefore $a = 1$ is an ordinary point, and every solution is analytic at $a = 1$ with $R \geq 1$.

We now "recenter" the coefficients in the ODE:

$$\begin{aligned} xy'' + y' + xy &= 0, \\ (x-1+1)y'' + y' + (x-1+1)y &= 0, \\ (x-1)y'' + y'' + y' + (x-1)y + y &= 0. \end{aligned}$$

Plugging in $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, we eventually obtain

$$(2a_2 + a_1 + a_0) + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1}\right)(x-1)^n = 0.$$

This gives the relations

$$a_2 = \frac{-(a_0 + a_1)}{2} \quad \text{and} \quad a_{n+2} = \frac{-(n+1)^2 a_{n+1} - a_n - a_{n-1}}{(n+2)(n+1)}$$

for $n \geq 1$.

With $a_0 = 1$, $a_1 = 0$ we find that

$$y_1(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{12}(x-1)^5 - \frac{13}{180}(x-1)^6 + \frac{13}{210}(x-1)^7 + \cdots,$$

and with $a_0 = 0$, $a_1 = 1$ we obtain

$$y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{3}{20}(x-1)^5 - \frac{1}{8}(x-1)^6 + \frac{271}{2520}(x-1)^7 \cdots.$$

By an earlier remark, the solution with $y(1) = 5$ and $y'(1) = -3$ is

$$\begin{aligned} y =& 5y_1 - 3y_2 \\ =& 5 - 3(x-1) - (x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 \\ &- \frac{1}{30}(x-1)^5 + \frac{1}{72}(x-1)^6 - \frac{11}{840}(x-1)^7 + \cdots. \end{aligned}$$