The Method of Frobenius

R. C. Daileda

Trinity University

Partial Differential Equations
Lecture 16

Motivating example

Failure of the power series method

Consider the ODE $2xy'' + y' + y = 0$. In standard form this is

$$y'' + \frac{1}{2x}y' + \frac{1}{2x}y = 0 \ \Rightarrow p(x) = q(x) = \frac{1}{2x}, \ g(x) = 0.$$

In exercise A.4.25 you showed that $1/x$ is analytic at any $a > 0$, with radius $R = a$. Hence:

> Every solution of $2xy'' + y' + y = 0$ is analytic at $a > 0$ with radius $R \geq a$ (i.e. given by a PS for $0 < x < 2a$).

However, since p, q, g are continuous for $x > 0$, general theory guarantees that:

> Every solution of $2xy'' + y' + y = 0$ is defined for *all* $x > 0$.

Question: Can we find series solutions defined for *all* $x > 0$?

Even though $p(x) = q(x) = 1/2x$ is *not* analytic at $a = 0$, we nonetheless assume

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{(with positive radius)}$$

and see what happens. Plugging into the ODE and collecting common powers of x leads to

$$a_{n+1} = \frac{-a_n}{(n+1)(2n+1)} \quad \text{for} \quad n \geq 1,$$

and then choosing $a_0 = 1$ yields the first solution

$$a_n = \frac{(-1)^n 2^n}{(2n)!} \quad \Rightarrow \quad y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n = \cos\left(\sqrt{2x}\right).$$

But choosing $a_0 = 0$ gives $a_n = 0$ for all $n \geq 0$, so that $y_2 \equiv 0$.

What now?

To find a second independent solution, we instead assume

$$y = x^r \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\text{PS with } R>0} = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (a_0 \neq 0)$$

for some $r \in \mathbb{R}$ *to be determined*. Since

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2},$$

plugging into the ODE gives

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Distributing the $2x$ and setting $m = n - 1$ in the first two series yields

$$\sum_{m=-1}^{\infty} 2(m+r+1)(m+r)a_{m+1}x^{m+r} + \sum_{m=-1}^{\infty} (m+1+r)a_{m+1}x^{m+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

or, replacing m with n

$$\underbrace{(2r(r-1)+r)a_0x^{r-1}}_{n=-1} + \sum_{n=0}^{\infty} \left((n+r+1)\left(2(n+r)+1\right)a_{n+1} + a_n\right)x^{n+r} = 0.$$

This requires the coefficients on each power of x to equal zero.

That is

$$r(2r-1)a_0 = 0 \underset{a_0 \neq 0}{\Rightarrow} r(2r-1) = 0 \Rightarrow r = 0, \frac{1}{2},$$

and $(n+r+1)(2n+2r+1)a_{n+1} + a_n = 0$, or

$$a_{n+1} = \frac{-a_n}{(n+r+1)(2n+2r+1)} \quad \text{for} \quad n \geq 0.$$

Each value of r gives a *different* recurrence:

$$r = 0 \Rightarrow a_{n+1} = \frac{-a_n}{(n+1)(2n+1)},$$

$$r = \frac{1}{2} \Rightarrow a_{n+1} = \frac{-a_n}{(n+3/2)(2n+2)} = \frac{-a_n}{(2n+3)(n+1)}.$$

Notice that the first is the original recurrence!

Taking $a_0 = 1$ in the second we eventually find that

$$a_n = \frac{(-1)^n 2^n}{(2n+1)!} \quad \Rightarrow \quad y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n = \frac{1}{\sqrt{2}} \sin\left(\sqrt{2x}\right).$$

This gives the second (linearly independent) solution to the ODE, and we have the general solution

$$y = c_1 y_1 + c_2 y_2 = c_1 \cos\left(\sqrt{2x}\right) + c_2' \sin\left(\sqrt{2x}\right) \quad (x > 0).$$

Remarks:

- The fact that both series yielded familiar functions is simply a coincidence, and should not be expected in general.
- One could also have obtained y_2 from y_1 (or vice-verse) using a technique called *reduction of order*.

Method of Frobenius - First Solution

When will the preceding technique work at an "extraordinary" point? Here's a *partial* answer:

Theorem

Suppose that at least one of $p(x)$ *or* $q(x)$ *is* not *analytic at* $x = 0$, *but that* both *of* $xp(x)$ *and* $x^2q(x)$ *are. If*

$$\lim_{x\to 0} xp(x) = p_0 \quad \textit{and} \quad \lim_{x\to 0} x^2q(x) = q_0,$$

then there is a solution to $y'' + p(x)y' + q(x)y = 0$ $(x > 0)$ *of the form*

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0),$$

where r *is a root of the* indicial equation $r^2 + (p_0 - 1)r + q_0 = 0$.

Remarks

- Under the hypotheses of the theorem, we say that $a = 0$ is a *regular singular point* of the ODE.
- Suppose the roots of the indicial equation are r_1 and r_2.
  - If $r_1 - r_2 \notin \mathbb{Z}$, then both $r = r_1$ and $r = r_2$ yield (linearly independent) solutions.
  - If $r_1 - r_2 \in \mathbb{Z}$, then only $r = \max\{r_1, r_2\}$ is *guaranteed* to work. The other may or *may not*.
- If the PS for $xp(x)$ and $x^2q(x)$ both converge for $|x| < R$, so does the PS factor of y.
- We can talk about regular singularities at any $x = a$ by instead considering $(x-a)p(x)$, $(x-a)^2q(x)$, $\lim\limits_{x \to a}$, and writing the solution in powers of $(x-a)$.

Example

Find the general solution to $x^2y'' + xy' + (x-2)y = 0$.

In standard form this ODE has

$$p(x) = \frac{1}{x} \quad \text{and} \quad q(x) = \frac{x-2}{x^2},$$

neither of which is analytic at $x = 0$. However, both

$$xp(x) = 1 \quad \text{and} \quad x^2 q(x) = x - 2$$

are analytic at $x = 0$, so we have a regular singularity with

$$p_0 = \lim_{x\to 0} xp(x) = 1 \quad \text{and } q_0 = \lim_{x\to 0} x^2 q(x) = -2.$$

The indicial equation is

$$r^2 + (1-1)r - 2 = 0 \;\Rightarrow\; r = \pm\sqrt{2}.$$

Applying the method of Frobenius, we set

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (a_0 \neq 0)$$

and substitute into the ODE, obtaining

$$(r^2 - 2)a_0 x^r + \sum_{n=1}^{\infty} \left(\left((n+r)^2 - 2 \right) a_n + a_{n-1} \right) x^{n+r} = 0.$$

Hence we must have $r^2 - 2 = 0$ (which we already knew) and

$$a_n = \frac{-a_{n-1}}{(n+r)^2 - 2} = \frac{-a_{n-1}}{n(n+2r)} \text{ for } n \geq 1.$$

Taking $a_0 = 1$ one readily sees that

$$a_n = \frac{(-1)^n}{n!(1+2r)(2+2r)(3+2r)\cdots(n+2r)}.$$

Since the difference of the roots is $\sqrt{2} - (-\sqrt{2}) = 2\sqrt{2} \notin \mathbb{Z}$, the two r-values give independent solutions:

$$y_1 = x^{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(1+2\sqrt{2})(2+2\sqrt{2})(3+2\sqrt{2})\cdots(n+2\sqrt{2})},$$
$$y_2 = x^{-\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(1-2\sqrt{2})(2-2\sqrt{2})(3-2\sqrt{2})\cdots(n-2\sqrt{2})},$$

and the general solution (for $x > 0$) is

$$y = c_1 y_1 + c_2 y_2.$$

Remark: Because $xp(x) = 1$ and $x^2 q(x) = x - 2$ both have infinite radius of convergence, so do both series above.

Method of Frobenius - Second Solution

What do we do if the indicial roots differ by an integer?

Theorem

Suppose that $x = 0$ is a regular singular point of $y'' + p(x)y' + q(x)y = 0$, and that the roots of the indicial equation are r_1 and r_2, with $r_1 - r_2 \in \mathbb{N}_0$.

- *If $r_1 = r_2 = r$, the second solution has the form*

$$y_2 = y_1 \ln x + x^r \sum_{n=1}^{\infty} b_n x^n.$$

- *If $r_1 > r_2$ (so that y_1 uses r_1), the second solution has the form*

$$y_2 = k y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad (b_0 \neq 0).$$

Example

Find the general solution to $xy'' + (1-x)y' + 2y = 0$, $x > 0$.

In standard form we have

$$p(x) = \frac{1-x}{x} \quad \text{and} \quad q(x) = \frac{2}{x},$$

which are non-analytic at $x = 0$, and

$$xp(x) = 1 - x \quad \text{and} \quad x^2 q(x) = 2x,$$

which are. This makes $x = 0$ a regular singularity with

$$p_0 = \lim_{x \to 0} 1 - x = 1 \quad \text{and} \quad \lim_{x \to 0} 2x = 0,$$

and indicial equation

$$r^2 + (1-1)r + 0 = 0 \quad \Rightarrow \quad r = 0.$$

Since $r = 0$ is a double root, we are guaranteed only one solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n.$$

Plugging this into the ODE and simplifying leads to the recursion

$$a_{n+1} = \frac{(n-2)a_n}{(n+1)^2} \text{ for } n \geq 0.$$

Taking $a_0 = 1$ we find that

$$a_1 = \frac{-2a_0}{1^2} = -2, \;\; a_2 = \frac{-a_1}{2^2} = \frac{1}{2}, \;\; a_3 = \frac{0 \cdot a_2}{3^2} = 0,$$

and hence $a_4 = a_5 = a_6 = \cdots = 0$ as well. So our first solution is

$$y_1 = 1 - 2x + \frac{x^2}{2}.$$

According to the theorem, a second independent solution has the form

$$y_2 = y_1 \ln x + \underbrace{x^0 \sum_{n=1}^{\infty} b_n x^n}_{w},$$

and we need to solve for the b_n. The product rule gives us

$$y_2' = y_1' \ln x + \frac{y_1}{x} + w',$$
$$y_2'' = y_1'' \ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2} + w'',$$

and plugging these into $xy_2'' + (1-x)y_2' + 2y_2 = 0$ we obtain

$$\underbrace{\left(xy_1'' + (1-x)y_1' + 2y_1\right)}_{=0} \ln x - y_1 + 2y_1' + xw'' + (1-x)w' + 2w = 0,$$

$$xw'' + (1-x)w' + 2w = -2y_1' + y_1.$$

We now plug $y_1 = 1 - 2x + x^2/2$ and $w = \sum_{n=1}^\infty b_n x^n$ into this equation to obtain a recurrence for the b_n:

$$b_1 + \sum_{n=1}^{\infty} \left((n+1)^2 b_{n+1} - (n-2)b_n\right) x^n = 5 - 4x + \frac{x^2}{2}.$$

Hence

$$b_1 = 5, \;\; 4b_2 + b_1 = -4, \;\; 9b_3 = \frac{1}{2},$$

and

$$b_{n+1} = \frac{(n-2)b_n}{(n+1)^2} \;\; \Rightarrow \;\; b_n = \frac{36 b_3}{n(n-1)(n-2)n!} \quad \text{for } n \geq 3.$$

Thus, since $b_3 = 1/18$,

$$y_2 = \underbrace{\left(1 - 2x + \frac{x^2}{2}\right)}_{y_1} \ln x + \underbrace{5x - \frac{9}{4}x^2 + 2\sum_{n=3}^{\infty} \frac{x^n}{n(n-1)(n-2)n!}}_{w}.$$

Finally, we have that the general solution is given by

$$y = c_1 y_1 + c_2 y_2.$$

Remarks. Regarding the case $r_1 - r_2 \in \mathbb{N}_0$:

- When y_1 has infinitely many nonzero coefficients, the general recursion for b_n will be more complicated.
- If a closed form expression for the coefficients of y_1 isn't available, the recursion relations for the a_n and b_n still allow us to compute as many terms as we need.
- Similar computations and comments hold when $r_1 - r_2 \in \mathbb{N}$, except that one must also solve for k.
- Because of the $\ln x$ factor, one can frequently conclude that $|y_2| \to \infty$ as $x \to 0^+$, without explicitly computing the b_n. This will suffice for our applications.