The Method of Characteristics

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Partial Differential Equations
Lecture 3

Linear and Quasi-Linear (first order) PDEs

A PDE of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C_1(x,y)u = C_0(x,y)$$

is called a (first order) *linear PDE* (in two variables). It is called *homogeneous* if $C_0 \equiv 0$.

More generally, a PDE of the form

$$A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u)$$

will be called a (first order) *quasi-linear PDE* (in two variables).

Remark: Every linear PDE is also quasi-linear since we may set

$$C(x,y,u) = C_0(x,y) - C_1(x,y)u.$$

Examples

Every PDE we saw last time was linear.

1. $\dfrac{\partial u}{\partial t} + v\dfrac{\partial u}{\partial x} = 0$ (the 1-D transport equation) is linear and homogeneous.
2. $5\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = x$ is linear and inhomogeneous.
3. $2y\dfrac{\partial u}{\partial x} + (3x^2 - 1)\dfrac{\partial u}{\partial y} = 0$ is linear and homogeneous.
4. $\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = u$ is linear and homogeneous.

Here are some quasi-linear examples.

5. $(x - y)\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = u^2$ is quasi-linear but *not* linear.
6. $\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = 0$ is quasi-linear but *not* linear.

Initial data

In addition to the PDE itself we will assume we are given the following additional information:

- A curve γ in the xy-plane on which the values of the solution $u(x, y)$ are specified, e.g.

$$u(x,0) = x^2 : \gamma \text{ is the } x\text{-axis;}$$
$$u(0,y) = ye^y : \gamma \text{ is the } y\text{-axis;}$$
$$u(x,x^3-x) = \sin x : \gamma \text{ is the graph of } y = x^3 - x.$$

- (Optional) The desired domain of the solution, e.g.

$$\{(x,y) \,|\, -\infty < x < \infty,\ \ y > 0\},$$
$$\{(x,y) \,|\, -\infty < x < y < \infty\}.$$

If one is not given, we seek the largest domain in the xy-plane possible.

A geometric approach

Goal: Develop a technique that will reduce any quasi-linear PDE

$$A(x,y,u)\frac{\partial u}{\partial x}+B(x,y,u)\frac{\partial u}{\partial y}=C(x,y,u) \quad \text{(plus initial data)}$$

to a system of ODEs.

Idea: Think geometrically. Identify the solution $u(x,y)$ with its graph, which is the surface in xyz-space defined by $z=u(x,y)$.

- The initial data along the curve γ gives us a space curve Γ that must lie on the graph. We call Γ the *initial curve* of the solution.
- We will use the PDE to build the remainder of the graph as a collection of additional space curves that "emanate from" Γ.

In Calc. 3, one learns that the normal vector to the surface $z = u(x, y)$ is

$$\mathbf{N} = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1 \right\rangle .$$

Let $\mathbf{F}$ denote the vector field

$$\mathbf{F} = \langle A(x, y, z), B(x, y, z), C(x, y, z) \rangle$$

defined by the coefficient functions in the given PDE.

Notice that if $u(x, y)$ solves the PDE, then on the surface $z = u(x, y)$ we have

$$\mathbf{F} \cdot \mathbf{N} = A(x, y, u)\frac{\partial u}{\partial x} + B(x, y, u)\frac{\partial u}{\partial y} - C(x, y, u) = 0.$$

Since

$$\begin{aligned} \mathbf{F}\cdot\mathbf{N} = 0 &\iff \mathbf{F} \text{ is perpendicular to } \mathbf{N} \\ &\iff \mathbf{F} \text{ is tangent to the graph } z = u(x,y), \end{aligned}$$

we see that:

- The graph of the solution $u(x, y)$ is made up of integral curves (stream lines) of the vector field $\mathbf{F}$.

Moral: *we can construct the graph of the solution to the PDE by finding the stream lines of* $\mathbf{F}$ *that pass through the initial curve* Γ.

This is equivalent to solving a system of ODEs!

The Method of Characteristics

Step 1. Parametrize the initial curve Γ, i.e. write

$$\Gamma : \begin{cases} x = x_0(a), \\ y = y_0(a), \\ z = z_0(a). \end{cases}$$

Step 2. For each a, find the stream line of $\mathbf{F}$ that passes through $\Gamma(a)$. That is, solve the system of ODE initial value problems

$$\frac{dx}{ds} = A(x,y,z), \quad \frac{dy}{ds} = B(x,y,z), \quad \frac{dz}{ds} = C(x,y,z),$$

$$x(0) = x_0(a), \qquad y(0) = y_0(a), \qquad z(0) = z_0(a).$$

These are the *characteristic equations* of the PDE.

The solutions to the system in **Step 2** will be in terms of the parameters a and s:

$$x = X(a,s), \quad y = Y(a,s), \tag{1}$$
$$z = Z(a,s). \tag{2}$$

This is a *parametric* expression for the graph of the solution surface $z = u(x,y)$ (in terms of the variables a, s).

Step 3. Solve the system (1) for a, s in terms of x, y:

$$a = \Lambda(x,y), \quad s = S(x,y).$$

Step 4. Substitute the results of **Step 3** into (2) to get the solution to the PDE:

$$u(x,y) = Z(\Lambda(x,y), S(x,y)).$$

Example

Find the solution to $x\dfrac{\partial u}{\partial x} - 2y\dfrac{\partial u}{\partial y} = u^2$ *that satisfies* $u(x,x) = x^3$.

This is a quasi-linear PDE with

$$A(x,y,u) = x, \ \ B(x,y,u) = -2y, \ \ C(x,y,u) = u^2,$$

so we may apply the method of characteristics.
The initial curve Γ can be parametrized as

$$x = a, \ \ y = a, \ \ z = a^3.$$

Hence the characteristic ODEs are

$$\frac{dx}{ds} = x, \qquad \frac{dy}{ds} = -2y, \qquad \frac{dz}{ds} = z^2,$$

$$x(0) = a, \quad y(0) = a, \qquad z(0) = a^3.$$

We find immediately that

$$x(s) = ae^s \quad \text{and} \quad y(s) = ae^{-2s}. \tag{3}$$

The equation in z is separable, with solution

$$z(s) = \frac{a^3}{1 - sa^3}. \tag{4}$$

We now need to solve (3) for a and s. We have $x/a = e^s$ so that

$$\begin{aligned} y = a(e^s)^{-2} = a(x/a)^{-2} = a^3/x^2 \quad &\Rightarrow \quad a^3 = x^2 y \\ &\Rightarrow \quad a = x^{2/3}y^{1/3}, \\ e^s = x/a = x^{1/3}y^{-1/3} = (x/y)^{1/3} \quad &\Rightarrow \quad s = \ln\left((x/y)^{1/3}\right) \\ &\qquad\quad = \frac{1}{3}\ln(x/y). \end{aligned}$$

Substituting these into (4) yields the solution to the PDE:

$$u(x, y) = \frac{x^2 y}{1 - \frac{1}{3}x^2 y \ln(x/y)}.$$

Remark. There are two main difficulties that can arise when applying this method:

- Solving the system of characteristic ODEs may be difficult (or impossible), especially if there is *coupling* between the equations.
- Passing from the parametric to the explicit form of the solution (i.e. solving for a and s in terms of x and y) may be difficult (or impossible), especially is the expressions for x and y are complicated.

Example

Find the solution to $(x-y)\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=x$ *that satisfies* $u(x,0)=f(x)$.

This PDE is linear, so quasi-linear. The initial curve is given by

$$x=a,\ \ y=0,\ \ z=f(a),$$

and so the characteristic ODEs are

$$\frac{dx}{ds}=x-y,\quad \frac{dy}{ds}=1,\quad \frac{dz}{ds}=x,$$

$$x(0)=a,\quad y(0)=0,\quad z(0)=f(a).$$

We see that $y(s) = s$, which means that the equation for x becomes

$$\frac{dx}{ds} = x - s \quad \text{or} \quad \frac{dx}{ds} - x = -s.$$

This is a linear ODE. Multiplying by the integrating factor e^{-s}, anti-differentiating, and using the initial condition $x(0) = a$ yields

$$x(s) = 1 + s + (a - 1)e^s.$$

This means that z satisfies

$$\frac{dz}{ds} = 1 + s + (a - 1)e^s \;\Rightarrow\; z(s) = s + \frac{s^2}{2} + (a - 1)(e^s - 1) + f(a),$$

since $z(0) = f(a)$.

Finally, we solve for a and s. We already have $s = y$ so that

$$x = 1 + s + (a-1)e^s = 1 + y + (a-1)e^y$$
$$\Rightarrow \quad a = 1 + (x - y - 1)e^{-y}.$$

Substituting these into the expression for z we obtain the solution to the PDE:

$$\begin{aligned} u(x,y) &= y + \frac{y^2}{2} + (x-y-1)e^{-y}(e^y - 1) + f\left(1 + (x-y-1)e^{-y}\right) \\ &= y + \frac{y^2}{2} + (x-y-1)(1-e^{-y}) + f\left(1 + (x-y-1)e^{-y}\right). \end{aligned}$$

Remark. When the PDE in question is linear:

- The characteristic ODEs for x and y will *never* involve z.
- The characteristic equation for z will *always* be a *linear* ODE.

Example

Find the solution to $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = e^u$ *that satisfies* $u(0, y) = y^2 - 1$.

This is a quasi-linear PDE with initial curve

$$x = 0, \;\; y = a, \;\; z = a^2 - 1,$$

and characteristic ODEs

$$\frac{dx}{ds} = y, \qquad \frac{dy}{ds} = -x, \qquad \frac{dz}{ds} = e^z,$$

$$x(0) = 0, \qquad y(0) = a, \qquad z(0) = a^2 - 1.$$

To decouple the first two equations, we differentiate again:

$$\frac{dx}{ds} = y \quad \Rightarrow \quad \frac{d^2x}{ds^2} = \frac{dy}{ds} = -x \quad \Rightarrow \quad \frac{d^2x}{ds^2} + x = 0.$$

This is a second order linear ODE with characteristic polynomial $r^2 + 1 = 0$, whose roots are $r = \pm i$. Consequently

$$x(s) = c_1 \cos s + c_2 \sin s \quad \Rightarrow \quad y(s) = x'(s) = -c_1 \sin s + c_2 \cos s.$$

From $x(0) = 0$ and $y(0) = a$ we obtain $c_1 = 0$ and $c_2 = a$, so that finally

$$x(s) = a \sin s, \quad y(s) = a \cos s.$$

Note that we immediately obtain

$$x^2 + y^2 = a^2 \quad \text{and} \quad \frac{x}{y} = \tan s.$$

The ODE for z is separable and solving it gives

$$z = -\ln\left(e^{1-a^2} - s\right).$$

Using the results of the previous slide, we find that the solution to the original PDE is

$$u(x,y) = -\ln\left(e^{1-x^2-y^2} - \arctan\left(\frac{x}{y}\right)\right).$$

Remark. We can think of the solutions to the first two characteristic ODEs

$$x = X(a,s), \quad y = Y(a,s)$$

as a change of coordinates. In the preceding example, we see that we have (essentially) switched to polar coordinates.

Example

Find the solution to $(u+2y)\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = 0$ *that satisfies* $u(x,1) = \dfrac{1}{x}$.

The initial curve can be parametrized by

$$x = a, \;\; y = 1, \;\; z = \frac{1}{a},$$

so that the characteristic ODEs are

$$\frac{dx}{ds} = z + 2y, \quad \frac{dy}{ds} = z, \quad \frac{dz}{ds} = 0,$$

$$x(0) = a, \qquad y(0) = 1, \quad z(0) = 1/a.$$

We solve for z first, obtaining $z(s) = 1/a$. The ODE for y then becomes

$$\left.\begin{array}{c} \dfrac{dy}{ds} = z = \dfrac{1}{a} \\[2ex] y(0) = 1 \end{array}\right\} \Rightarrow \quad y(s) = \frac{s}{a} + 1.$$

Finally, we substitute these into the ODE for x:

$$\left.\begin{array}{c} \dfrac{dx}{ds} = z + 2y = \dfrac{1}{a} + \dfrac{2s}{a} + 2 \\[2ex] x(0) = a \end{array}\right\} \Rightarrow \quad x(s) = \frac{s}{a} + \frac{s^2}{a} + 2s + a.$$

To get the solution to the PDE, we need to express a in terms of x and y.

From $y = s/a + 1$ we have $s = a(y-1)$. We plug this into the expression for x:

$$\begin{aligned} x &= \frac{s}{a} + \frac{s^2}{a} + 2s + a \\ &= \frac{a(y-1)}{a} + \frac{(a(y-1))^2}{a} + 2a(y-1) + a \\ &= y - 1 + a((y-1)^2 + 2(y-1) + 1) \\ &= y - 1 + ay^2. \end{aligned}$$

So $a = \dfrac{x - y + 1}{y^2}$ and (since $z = 1/a$) the solution to the PDE is

$$u(x,y) = \frac{y^2}{x - y + 1}.$$