Linear PDEs and the Principle of Superposition

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Partial Differential Equations
Lecture 8
**Definition:** A *linear differential operator* (in the variables $x_1, x_2, \ldots, x_n$) is a sum of terms of the form

$$A(x_1, x_2, \ldots, x_n) \frac{\partial^{a_1+a_2+\cdots+a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}},$$

where each $a_i \geq 0$.

**Examples:** The following are linear differential operators.

1. The Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

2. $W = c^2 \nabla^2 - \frac{\partial^2}{\partial t^2}$
3. \( H = c^2 \nabla^2 - \frac{\partial}{\partial t} \)

4. \( T = \frac{\partial}{\partial t} - v_1 \frac{\partial}{\partial x_1} - v_2 \frac{\partial}{\partial x_2} - \cdots - v_n \frac{\partial}{\partial x_n} = \frac{\partial}{\partial t} - \mathbf{v} \cdot \nabla \)

5. The general first order linear operator (in two variables):

\[
D_1 = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} + C(x, y)
\]

6. The general second order linear operator (in two variables):

\[
D_2 = A(x, y) \frac{\partial^2}{\partial x^2} + 2B(x, y) \frac{\partial^2}{\partial x \partial y} + C(x, y) \frac{\partial^2}{\partial y^2} \\
+ D(x, y) \frac{\partial}{\partial x} + E(x, y) \frac{\partial}{\partial y} + F(x, y)
\]
Theorem

If $D$ is a linear differential operator (in the variables $x_1, x_2, \cdots, x_n$), $u_1$ and $u_2$ are functions (in the same variables), and $c_1$ and $c_2$ are constants, then

$$D(c_1 u_1 + c_2 u_2) = c_1 Du_1 + c_2 Du_2.$$ 

Remarks:

- This follows immediately from the fact that each partial derivative making up $D$ has this property, e.g.

$$\frac{\partial^3}{\partial x_1^2 \partial x_2} (c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} + c_2 \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}.$$ 

- This property extends (in the obvious way) to any number of functions and constants.
**Definition:** A linear PDE (in the variables $x_1, x_2, \cdots, x_n$) has the form

$$Du = f$$  \hspace{1cm} (1)

where:

- $D$ is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- $f$ is a function (of $x_1, x_2, \cdots, x_n$).

We say that (1) is *homogeneous* if $f \equiv 0$. 
Examples

The following are linear PDEs.

1. The Laplace equation: $\nabla^2 u = 0$ (homogeneous)

2. The wave equation: $c^2 \nabla^2 u - \frac{\partial^2 u}{\partial t^2} = 0$ (homogeneous)

3. The heat equation: $c^2 \nabla^2 u - \frac{\partial u}{\partial t} = 0$ (homogeneous)

4. The Poisson equation: $\nabla^2 u = f(x_1, x_2, \ldots, x_n)$
   (inhomogeneous if $f \not\equiv 0$)

5. The advection equation: $\frac{\partial u}{\partial t} + \kappa \frac{\partial u}{\partial x} + ru = k(x, t)$
   (inhomogeneous if $k \not\equiv 0$)

6. The telegraph equation: $\frac{\partial^2 u}{\partial t^2} + 2B \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} + Au = 0$
   (homogeneous)
A boundary value problem (BVP) consists of:

- a domain $\Omega \subseteq \mathbb{R}^n$,
- a PDE (in $n$ independent variables) to be solved in the interior of $\Omega$,
- a collection of boundary conditions (BCs) to be satisfied on the boundary of $\Omega$.

The data for a BVP:
Definition: Let $\Omega \subseteq \mathbb{R}^n$ be the domain of a BVP and let $A$ be a subset of the boundary of $\Omega$.

We say that a BC on $A$ is \textit{linear} if it has the form

$$\delta u|_A = f|_A \tag{2}$$

where:

- $\delta$ is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- $f$ is a function (of $x_1, x_2, \cdots, x_n$).

(The notation $\cdot|_A$ means “restricted to $A$.”) We say that (2) is \textit{homogeneous} if $f \equiv 0$. 
Examples

The following are linear BCs.

1. *Dirichlet conditions*: \( u\big|_A = f\big|_A \), such as

\[
    u(x, 0) = f(x) \text{ for } 0 < x < L, \text{ or } u(L, t) = 0 \text{ for } t > 0
\]

2. *Neumann conditions*: \( \left. \frac{\partial u}{\partial n} \right|_A = f\big|_A \), where \( \frac{\partial u}{\partial n} \) is the directional derivative perpendicular to \( A \), such as

\[
    u_t(x, 0) = g(x) \text{ for } 0 < x < L, \text{ or } u_x(0, t) = 0 \text{ for } t > 0
\]

3. *Robin conditions*: \( u + a \left. \frac{\partial u}{\partial n} \right|_A = f\big|_A \), such as

\[
    u(L, t) + u_x(L, t) = 0 \text{ for } t > 0
\]
The principle of superposition

Let $D$ and $\delta$ be linear differential operators (in the variables $x_1, x_2, \ldots, x_n$), let $f_1$ and $f_2$ be functions (in the same variables), and let $c_1$ and $c_2$ be constants.

- If $u_1$ solves the linear PDE $Du = f_1$ and $u_2$ solves $Du = f_2$, then $u = c_1u_1 + c_2u_2$ solves $Du = c_1f_1 + c_2f_2$. In particular, if $u_1$ and $u_2$ both solve the same homogeneous linear PDE, so does $u = c_1u_1 + c_2u_2$.

- If $u_1$ satisfies the linear BC $\delta u|_A = f_1|_A$ and $u_2$ satisfies $\delta u|_A = f_2|_A$, then $u = c_1u_1 + c_2u_2$ satisfies $\delta u|_A = c_1f_1 + c_2f_2|_A$. In particular, if $u_1$ and $u_2$ both satisfy the same homogeneous linear BC, so does $u = c_1u_1 + c_2u_2$. 
Remarks on the superposition principle

- It is an easy consequence of the linearity of $D, \delta$, e.g. if $Du_1 = f_1$ and $Du_2 = f_2$, then
  \[ D(c_1 u_1 + c_2 u_2) = c_1 Du_1 + c_2 Du_2 = c_1 f_1 + c_2 f_2. \]

- It extends (in the obvious way) to any number of functions and constants.

- It implies that linear combinations of functions that satisfy homogeneous linear PDEs/BCs satisfy \textit{the same equations}. 
**Warning:** The principle of superposition can *easily* fail for nonlinear PDEs or boundary conditions.

Consider the nonlinear PDE

\[ u_x + u^2 u_y = 0. \]

One solution of this PDE is

\[ u_1(x, y) = \frac{-1 + \sqrt{1 + 4xy}}{2x}. \]

However, the function \( u = cu_1 \) *does not* solve the same PDE unless \( c = 0, \pm 1 \).
Superposition and separation of variables

Consider a linear BVP consisting of the following data:

(A) A *homogeneous* linear PDE on a region $\Omega \subseteq \mathbb{R}^n$;
(B) A (finite) list of *homogeneous* linear BCs on (part of) $\partial \Omega$;
(C) A (finite) list of *inhomogeneous* linear BCs on (part of) $\partial \Omega$.

Roughly speaking, to solve such a problem one:

1. Finds all “separated” solutions to (A) and (B).
   - This amounts to solving a collection of linear ODE BVPs linked by separation constants.
   - Superposition guarantees *any linear combination* of separated solutions also solves (A) and (B).

2. Determines the specific linear combination of separated solutions that solves (C).
Remarks on separation of variables

- When separated solutions involve sines and cosines, finding the solutions to inhomogeneous BCs utilize Fourier series/half-range expansions.

- More generally, one must make use of “Fourier like” series involving other families of orthogonal functions (e.g. Sturm-Liouville theory).

- When there are no homogeneous BCs, or “too many” inhomogeneous BCs, one can “homogenize” parts of the problem and then superimpose these partial results to get the complete solution.

- Depending on the shape of the domain in question, successful separation of variables may require change of coordinates.