Math 3357 Spring 2015

Partial Differential Equations

First Midterm Exam

Thursday, February 12

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**
Points	17	18	15	20	10	20
Score						

Total:______________________

1. Consider the PDE

$$3\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = (x+y)u^2 \tag{1}$$

a. Find constants c and d so that the linear change of variables

$$\alpha = x + by,$$
$$\beta = x + dy,$$

transforms (1) into a PDE involving *only one* partial derivative.

b. Find the general solution to the transformed PDE you found in part **a**. [*Remember:* α "thinks" β is a constant, and vice versa.]

c. Find the general solution to (1) by writing the solution you found in part **c** in terms of x and y.

2. Consider the initial value problem

$$\begin{aligned} & x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2, \\ & u(y^3, y) = y^2. \end{aligned} \tag{2}$$

a. Write down the system of characteristic ODEs of (2).

b. Solve the characteristic system you found in part **a**.

c. Use the result of part **b** to find the solution to (2).

3. Recall that the *Lapace equation* in two variables is the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{3}$$

a. Show that any function of the form

$$u(x, y) = F(x + iy) + G(x - iy),$$

where F and G are twice-differentiable and $i^2 = -1$, is a solution of the Laplace equation.

b. Find functions F and G in part **a** so that $u(x, y)$ solves the *Dirichlet problem*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \;\; u(x, 0) = e^{-x}, \;\; u_y(x, 0) = 2,$$

on the domain $H = \mathbb{R} \times [0, \infty)$. [*Remark:* Your answer may involve i.]

c. Show that the function $u(x, y)$ of part **b** is actually real-valued, i.e. show that all imaginary quantities in your answer cancel out.

4. Consider the 4-periodic function satisfying

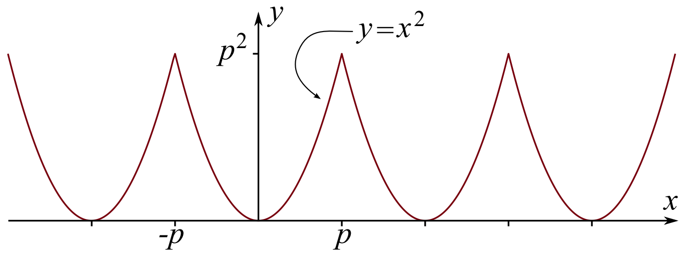
$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0, \\ 3 - x & \text{if } 0 \le x < 3. \end{cases}$$

a. Carefully sketch the graph of the Fourier series of f (for at least three periods). Be sure to include labeled axes and indicate the scale on each.

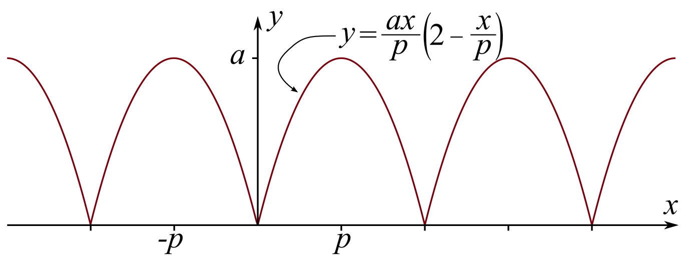
b. Find the Fourier series of f.

5. Let $f(x)$ and $g(x)$ be the $2p$-periodic functions whose graphs are shown below.

Graph of $f(x)$:



Graph of $g(x)$:



a. List the geometric operations necessary to transform the graph of $f(x)$ into the graph of $g(x)$.

b. The Fourier series of $f(x)$ is

$$\frac{p^2}{3} + \frac{4p^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{p}\right).$$

Use this and your answer to part **a** to find the Fourier series of $g(x)$ *without using Euler's integral formulas.*

6. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

a. If $m \neq n$, show that the inner product of $\sin(mx)$ and $\cos(nx)$ on the interval $[0, \pi]$ equals

$$\frac{m\left((-1)^{m+n+1}+1\right)}{m^2-n^2}.$$

b. Compute the inner product of $\sin(mx)$ and $\cos(mx)$ on the interval $[0, \pi]$.

c. Use parts **b** and **c** to show that $\sin(mx)$ and $\cos(nx)$ are orthogonal on the interval $[0, \pi]$ if and only if m and n have the *same parity*, i.e. are either both even or both odd.

d. How does the result of part **c** compare with the situation when we use the interval $[-\pi, \pi]$ instead?

Useful Trigonometric Identities.

$e^{iA} = \cos A + i \sin A \quad (i^2 = -1)$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$

$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$