Math 3357 Spring 2014

Partial Differential Equations

Second Midterm Exam

Thursday, March 27

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ________________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**
Points	18	12	12	20	12	26
Score						

Total:______________________

1. For $0 \le x \le 2$, let

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ x - 1 & \text{if } 1 \le x \le 2, \end{cases} \qquad \text{and} \quad g(x) = 2 - x.$$

Carefully sketch the following functions, for at least 3 periods. Be sure to include and label your axes, and pay careful attention to the values at any points of discontinuity.

a. The cosine series of $f(x)$.

b. The sine series of $f(x)$.

c. The cosine series of $g(x)$.

d. The sine series of $g(x)$.

2. Let f denote the 2π-periodic function satisfying $f(x) = |x|$ for $-\pi \le x \le \pi$. One can show that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

a. Let $g(x) = f\left(x + \frac{\pi}{2}\right) - \frac{\pi}{2}$. Use the given Fourier series of f and the identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$ to obtain

$$g(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \sin((2k+1)x)}{(2k+1)^2}.$$

b. Solve the heat problem

$$\begin{aligned} &u_t = 2u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ &u(0,t) = u(\pi,t) = 0, \quad t > 0, \\ &u(x,0) = \begin{cases} \frac{200}{\pi} x & \text{if } 0 \le x \le \frac{\pi}{2}, \\ \frac{200}{\pi}(\pi - x) & \text{if } \frac{\pi}{2} \le x \le \pi. \end{cases} \end{aligned}$$

[*Suggestion:* Determine how $u(x,0)$ is related to $g(x)$.]

3. Use separation of variables to reduce the PDE

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t}$$

to a system of three ODEs. **Do not attempt to solve this system.**

4. Show that the solution to the ODE boundary value problem

$$
\begin{aligned}
&X'' - kX = 0, \\
&X(0) - X'(0) = 0, \\
&X(1) = 0,
\end{aligned}
$$

is given by $X = \mu\cos(\mu x) + \sin(\mu x)$, where μ satisfies $\tan\mu = -\mu$.

5. In the vibrating rectangular membrane problem, suppose that $a = 1$, $b = 2$, $c = 1/\pi$ and

$$\begin{aligned} f(x, y) &= -\sin(\pi x)\sin(2\pi y), \\ g(x, y) &= 3\sin(2\pi x)\sin(\pi y) - 4\sin(3\pi x)\sin(2\pi y). \end{aligned}$$

Find an explicit expression for $u(x, y, t)$ *without* any integral computations. [*Suggestion:* Plug $t = 0$ into u and u_t and compare the resulting expressions with f and g, respectively.]

6. Consider the following Dirichlet problem on a semicircle of radius a:

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < a, \quad 0 < \theta < \pi, \\
&u(r,0) = u(r,\pi) = 0, \quad 0 \le r < a, \\
&u(a,\theta) = f(\theta), \quad 0 \le \theta \le \pi.
\end{aligned}
$$

a. Provide a physical interpretation of this problem. What does $u(r,\theta)$ represent? What do the boundary conditions represent?

b. Find the separated solutions of the PDE that satisfy the homogeneous boundary conditions. [*Remark:* You should recognize one of the separated ODE boundary value problems. Feel free to simply cite its solution.]

b. (continued)

c. Use the principle of superposition to find the general solution to the entire Dirichlet problem. Give explicit formulas for any unknown coefficients in your solution.

- The solution of the boundary value problem

$$\begin{aligned} &u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ &u(0,t) = u(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < L, \end{aligned}$$

is $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(\mu_n x)$, where $\mu_n = \frac{n\pi}{L}$, $\lambda_n = c\mu_n$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\mu_n x)\,dx.$$

- The solution of the boundary value problem

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ &u(0,t) = u(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L, \end{aligned}$$

is $u(x,t) = \sum_{n=1}^{\infty} \left(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)\right) \sin(\mu_n x)$, where $\mu_n = \frac{n\pi}{L}$, $\lambda_n = c\mu_n$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\mu_n x)\,dx, \quad b_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin(\mu_n x)\,dx.$$

- The solution of the boundary value problem

$$\begin{aligned} &u_t = c^2(u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ &u(0,y,t) = u(a,y,t) = 0, \quad 0 \le y \le b, \quad t \ge 0, \\ &u(x,0,t) = u(x,b,t) = 0, \quad 0 \le x \le a, \quad t \ge 0, \\ &u(x,y,0) = f(x,y), \quad 0 < x < a, \quad 0 < y < b, \end{aligned}$$

is $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(\mu_m x)\sin(\nu_n y) e^{-\lambda_{mn}^2 t}$, where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, and

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin(\mu_m x)\sin(\nu_n y)\,dx\,dy.$$

- The solution of the boundary value problem

$$\begin{aligned} &u_{tt} = c^2(u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ &u(0,y,t) = u(a,y,t) = 0, \quad 0 \le y \le b, \quad t \ge 0, \\ &u(x,0,t) = u(x,b,t) = 0, \quad 0 \le x \le a, \quad t \ge 0, \\ &u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y), \quad 0 < x < a, \quad 0 < y < b, \end{aligned}$$

is $u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(B_{mn}\cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t)\right)\sin(\mu_m x)\sin(\nu_n y)$, where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, and

$$\begin{aligned} B_{mn} &= \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin(\mu_m x)\sin(\nu_n y)\,dx\,dy, \\ B_{mn}^* &= \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin(\mu_m x)\sin(\nu_n y)\,dx\,dy. \end{aligned}$$

- The solution of the boundary value problem

$$\begin{aligned}
&u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b, \quad t>0,\\
&u(x,0)=f_1(x), \quad u(x,b)=f_2(x), \quad 0\le x\le a,\\
&u(0,y)=g_1(y), \quad u(a,y)=g_2(y), \quad 0\le x\le b,
\end{aligned}$$

  is

$$\begin{aligned}
u(x,y)=&\sum_{n=1}^{\infty}A_n\sin(\mu_n x)\sinh(\mu_n(b-y))+\sum_{n=1}^{\infty}B_n\sin(\mu_n x)\sinh(\mu_n y)\\
&+\sum_{n=1}^{\infty}C_n\sinh(\nu_n(a-x))\sin(\nu_n y)+\sum_{n=1}^{\infty}D_n\sinh(\nu_n x)\sinh(\nu_n y),
\end{aligned}$$

  where $\mu_n=\frac{n\pi}{a}$, $\nu_n=\frac{n\pi}{b}$ and

$$\begin{aligned}
A_n&=\frac{2}{a\sinh\mu_n b}\int_0^a f_1(x)\sin(\mu_n x)\,dx, \quad B_n=\frac{2}{a\sinh\mu_n b}\int_0^a f_2(x)\sin(\mu_n x)\,dx,\\
C_n&=\frac{2}{b\sinh\nu_n a}\int_0^b g_1(y)\sin(\nu_n y)\,dy, \quad D_n=\frac{2}{b\sinh\nu_n a}\int_0^b g_2(y)\sin(\nu_n y)\,dy.
\end{aligned}$$

- The general solution of the boundary value problem

$$\begin{aligned}
&u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, \quad 0<r<a, \quad 0<\theta<2\pi,\\
&u(a,\theta)=f(\theta), \quad 0\le\theta\le 2\pi,
\end{aligned}$$

  is $u(r,\theta)=a_0+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(a_n\cos(n\theta)+b_n\sin(n\theta))$, where a_0, a_n and b_n are the Fourier coefficients of the 2π-periodic function $f(\theta)$.

- The general solution to the ODE $x^2y''+\alpha xy'+\beta y=0$ is

$$y=\begin{cases} c_1x^{\rho_1}+c_2x^{\rho_2} & \text{if } \rho_1\ne\rho_2,\\ c_1x^{\rho_1}+c_2x^{\rho_1}\ln x & \text{if } \rho_1=\rho_2,\end{cases}$$

  where ρ_1 and ρ_2 are the roots of $\rho^2+(\alpha-1)\rho+\beta=0$.