Math 3357 Spring 2015

Partial Differential Equations

Second Midterm Exam

Tuesday, March 31

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ___________________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**
Points	16	15	20	15	16	18
Score						

Total:________________________

1. For $0 \le x \le 3$ let

$$f(x) = x - 1 \quad \text{and} \quad g(x) = \begin{cases} 2 - x & \text{if } 0 \le x < 2, \\ 0 & \text{if } 2 < x \le 3. \end{cases}$$

Carefully sketch the following functions, for at least 3 periods. Be sure to include and label your axes, and pay careful attention to the values at any points of discontinuity.

a. The cosine series of $f(x)$.

b. The sine series of $f(x)$.

c. The cosine series of $g(x)$.

d. The sine series of $g(x)$.

2. Use separation of variables to reduce the PDE

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta} + \csc^2\theta\frac{\partial^2 u}{\partial \phi^2}\right) = 0$$

to a system of three ODEs. **Do not attempt to solve this system.**

3. Consider the ODE boundary value problem

$$\begin{aligned} &X'' + kX = 0, \quad 0 < x < 1, \\ &X'(0) = 0, \quad X'(1) = X(1). \end{aligned} \tag{1}$$

a. Show that if $k = 0$, the only solution to (1) is $X \equiv 0$.

b. Show that if $k = -\mu^2 < 0$, then (up to scalar multiples) the only solutions to (1) are $X = e^{\mu x} + e^{-\mu x}$, where $e^{2\mu} = \dfrac{\mu + 1}{\mu - 1}$.

c. Show that if $k = \mu^2 > 0$, then (up to scalar multiples) the only solutions to (1) are $X = \cos(\mu x)$, where $\tan \mu = -1/\mu$.

4. Consider the PDE boundary value problem

$$\begin{aligned} &u_t = c^2 u_{xx}, \quad t > 0, \quad 0 < x < L, \\ &u(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < L. \end{aligned} \tag{2}$$

One can show that the separated solutions of (the homogeneous portion of) (2) are given by

$$u_n(x,t) = \sin\left(\frac{(2n+1)\pi x}{2L}\right) e^{-\left(\frac{(2n+1)c\pi}{2L}\right)^2 t}, \quad n \in \mathbb{N}_0,$$

and that the functions $X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right)$, $n \in \mathbb{N}_0$, are pairwise orthogonal on $[0, L]$.

a. Use the Principal of Superposition to express the general solution to (2) as a series.

b. Compute the inner product (on $[0, L]$) of $X_n(x)$ with itself . [*Suggestion:* Use the identity $2\sin^2 A = 1 - \cos(2A)$.]

c. Express the coefficients in your answer to **a** in the form (constant)·(integral involving f).

5. If a constant (vertical) gravitational acceleration $g > 0$ is included in the derivation of the vibrating string problem, one is led to the PDE boundary value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - g, \quad t > 0, \quad 0 < x < L, \\ &u(0,t) = u(L,t) = 0, \quad t > 0, \end{aligned} \tag{3}$$

for the displacement $u(x,t)$ of the string from the x-axis at time t.

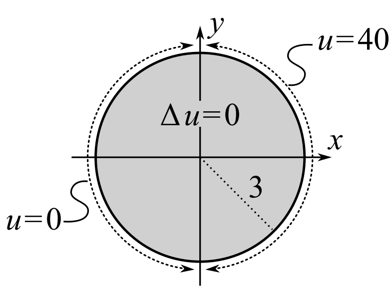
a. Determine the steady state shape of the string (i.e. when $u_t \equiv 0$).

b. If the string is initially given the shape of the function $f(x) = 0$ and released with no initial velocity, determine its shape at any later time. [*Suggestions:* "Homogenize" the PDE boundary value problem in question using your answer to part **a**. You can avoid all integral computations by making use of the given table of Fourier series.]

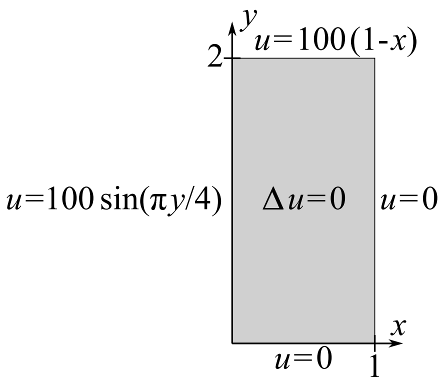
b. (continued)

6. Solve the Dirichlet problem described by the diagram. [*Suggestion:* You can avoid all integral computations by making use of the given table of Fourier series.]

a.



b.



- The solution of the boundary value problem

$$u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = u(L,t) = 0, \quad t > 0,$$
$$u(x,0) = f(x), \quad 0 < x < L,$$

is $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(\mu_n x)$, where $\mu_n = \frac{n\pi}{L}$, $\lambda_n = c\mu_n$ and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin(\mu_n x)\, dx.$$

- The solution of the boundary value problem

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = u(L,t) = 0, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L,$$

is $u(x,t) = \sum_{n=1}^{\infty} (b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)) \sin(\mu_n x)$, where $\mu_n = \frac{n\pi}{L}$, $\lambda_n = c\mu_n$ and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin(\mu_n x)\, dx, \quad b_n^* = \frac{2}{L\lambda_n} \int_0^L g(x) \sin(\mu_n x)\, dx.$$

- The solution of the boundary value problem

$$u_t = c^2 (u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$
$$u(0,y,t) = u(a,y,t) = 0, \quad 0 \le y \le b, \quad t \ge 0,$$
$$u(x,0,t) = u(x,b,t) = 0, \quad 0 \le x \le a, \quad t \ge 0,$$
$$u(x,y,0) = f(x,y), \quad 0 < x < a, \quad 0 < y < b,$$

is $u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(\mu_m x) \sin(\nu_n y) e^{-\lambda_{mn}^2 t}$, where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, and

$$A_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin(\mu_m x) \sin(\nu_n y)\, dx\, dy.$$

- The solution of the boundary value problem

$$u_{tt} = c^2 (u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$
$$u(0,y,t) = u(a,y,t) = 0, \quad 0 \le y \le b, \quad t \ge 0,$$
$$u(x,0,t) = u(x,b,t) = 0, \quad 0 \le x \le a, \quad t \ge 0,$$
$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y), \quad 0 < x < a, \quad 0 < y < b,$$

is $u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (B_{mn} \cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t)) \sin(\mu_m x) \sin(\nu_n y)$, where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, and

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin(\mu_m x) \sin(\nu_n y)\, dx\, dy,$$
$$B_{mn}^* = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x,y) \sin(\mu_m x) \sin(\nu_n y)\, dx\, dy.$$

- The solution of the boundary value problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\ &u(x,0) = f_1(x), \quad u(x,b) = f_2(x), \quad 0 \le x \le a, \\ &u(0,y) = g_1(y), \quad u(a,y) = g_2(y), \quad 0 \le x \le b, \end{aligned}$$

  is

$$\begin{aligned} u(x,y) = &\sum_{n=1}^{\infty} A_n \sin(\mu_n x)\sinh(\mu_n(b-y)) + \sum_{n=1}^{\infty} B_n \sin(\mu_n x)\sinh(\mu_n y) \\ &+ \sum_{n=1}^{\infty} C_n \sinh(\nu_n(a-x))\sin(\nu_n y) + \sum_{n=1}^{\infty} D_n \sinh(\nu_n x)\sinh(\nu_n y), \end{aligned}$$

  where $\mu_n = \frac{n\pi}{a}$, $\nu_n = \frac{n\pi}{b}$ and

$$\begin{aligned} A_n &= \frac{2}{a\sinh(\mu_n b)} \int_0^a f_1(x)\sin(\mu_n x)\,dx, \quad B_n = \frac{2}{a\sinh(\mu_n b)} \int_0^a f_2(x)\sin(\mu_n x)\,dx, \\ C_n &= \frac{2}{b\sinh(\nu_n a)} \int_0^b g_1(y)\sin(\nu_n y)\,dy, \quad D_n = \frac{2}{b\sinh(\nu_n a)} \int_0^b g_2(y)\sin(\nu_n y)\,dy. \end{aligned}$$

- The general solution of the boundary value problem

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < a, \quad 0 < \theta < 2\pi, \\ &u(a,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi, \end{aligned}$$

  is $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta))$, where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta, \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \quad (n > 0), \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta \quad (n > 0). \end{aligned}$$

Function	**Fourier Series**
$f(x)=\begin{cases}1 & \text{if } 0\le x<p,\\ -1 & \text{if } -p\le x<0\end{cases}$	$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi x}{p}\right)$
$f(x)=x \text{ if } -p\le x<p$	$\frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{p}\right)$
$f(x)=x^2 \text{ if } -p\le x<p$	$\frac{p^2}{3}-\frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{p}\right)$
$f(x)=p-\lvert x\rvert \text{ if } -p\le x<p$	$\frac{p}{2}+\frac{4p}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\left(\frac{(2k+1)\pi x}{p}\right)$
$f(x)=\begin{cases}p-x & \text{if } 0\le x<p,\\ -p-x & \text{if } -p\le x<0\end{cases}$	$\frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{p}\right)$
$f(x)=\begin{cases}1 & \text{if } \lvert x\rvert<\frac{p}{2},\\ 0 & \text{if } \lvert x\rvert\ge\frac{p}{2}\end{cases}$	$\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos\left(\frac{n\pi x}{p}\right)$
$f(x)=\begin{cases}1 & \text{if } 0\le x<\frac{p}{2},\\ -1 & \text{if } \frac{-p}{2}<x<0,\\ 0 & \text{if } \lvert x\rvert\ge\frac{p}{2}\end{cases}$	$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}\sin\left(\frac{n\pi x}{p}\right)$
$f(x)=x(p-\lvert x\rvert) \text{ if } -p\le x<p$	$\frac{8p^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\sin\left(\frac{(2k+1)\pi x}{p}\right)$
$f(x)=\lvert x\rvert(p-\lvert x\rvert) \text{ if } -p\le x<p$	$\frac{p^2}{6}-\frac{p^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos\left(\frac{2k\pi x}{p}\right)$
$f(x)=\begin{cases}\lvert x\rvert & \text{if } \lvert x\rvert\le\frac{p}{2},\\ \frac{p}{2} & \text{if } \lvert x\rvert>\frac{p}{2}\end{cases}$	$\frac{3p}{8}+\frac{2p}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)-1}{n^2}\cos\left(\frac{n\pi x}{p}\right)$
$f(x)=\begin{cases}x & \text{if } \lvert x\rvert\le\frac{p}{2},\\ \frac{p}{2} & \text{if } x>\frac{p}{2},\\ \frac{-p}{2} & \text{if } x<\frac{-p}{2}\end{cases}$	$\frac{p}{\pi^2}\sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{2}\right)+(-1)^{n+1}n\pi}{n^2}\sin\left(\frac{n\pi x}{p}\right)$
$f(x)=\begin{cases}\lvert x\rvert & \text{if } \lvert x\rvert\le\frac{p}{2},\\ 0 & \text{if } \lvert x\rvert>\frac{p}{2}\end{cases}$	$\frac{p}{8}+\frac{p}{\pi^2}\sum_{n=1}^{\infty}\frac{2\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)n\pi-2}{n^2}\cos\left(\frac{n\pi x}{p}\right)$
$f(x)=\begin{cases}x & \text{if } \lvert x\rvert\le\frac{p}{2},\\ 0 & \text{if } \lvert x\rvert>\frac{p}{2}\end{cases}$	$\frac{p}{\pi^2}\sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{2}\right)-\cos\left(\frac{n\pi}{2}\right)n\pi}{n^2}\sin\left(\frac{n\pi x}{p}\right)$
$f(x)=\left\lvert\sin\left(\frac{\pi x}{2p}\right)\right\rvert \text{ if } -p\le x<p$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\left(\frac{n\pi x}{p}\right)$
$f(x)=\sin\left(\frac{\pi x}{2p}\right) \text{ if } -p\le x<p$	$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{4n^2-1}\sin\left(\frac{n\pi x}{p}\right)$