Math 3357 Spring 2014

Partial Differential Equations

Third Midterm Exam

Tuesday, April 29

Your name (please print):

Instructions: This is an open book, open notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**
Points	20	20	20	20	20
Score					

Total:________________________

1. An ideal elastic membrane with a radius of 2 units is set in motion at $t = 0$ with an initial shape described by the graph of the function $f(r, \theta) = (4 - r^2)r^3 \sin(3\theta)$, and an initial uniform upward velocity of 1 unit per second. If the membrane constant is $c = 1$, find an expression for the shape of the membrane at any later time.

2. Consider the ODE boundary value problem

$$\begin{aligned} &y'' + 2y' + y + \lambda(x+1)^2 e^{-2x} y = 0, \quad 0 < x < 1, \\ &y(0) = 0, \quad y'(1) = 0. \end{aligned} \tag{1}$$

a. Put (1) in Sturm-Liouville form. Identify $p(x)$, $q(x)$ and $r(x)$. [*Suggestion:* Multiply by e^{2x}.]

b. Give an expression for the inner product $\langle f, g \rangle$ associated with (1).

c. Show that eigenfunctions of (1) with distinct eigenvalues are always orthogonal (relative to the inner product in **b**).

d. Show that $\lambda = 0$ is an eigenvalue of (1).

3. Consider the ODE boundary value problem

$$\begin{aligned} &y'' + (1 - x^2)y + \lambda y = 0, \quad 0 < x < \infty, \\ &y'(0) = 0, \quad \lim_{x \to \infty} y(x) = 0. \end{aligned} \tag{2}$$

a. Put (2) in Sturm-Liouville form. Identify $p(x)$, $q(x)$ and $r(x)$.

b. Give an expression for the inner product $\langle f, g \rangle$ associated with (2).

c. Show that $y = e^{-x^2/2}$ and $y = e^{-x^2/2}(4x^2 - 2)$ are eigenfunctions of (2).

d. Show that $\int_0^\infty (4x^2 - 2)e^{-x^2}\, dx = 0$. [*Suggestion:* Use parts **b** and **c**.]

4. The surface of a solid sphere of radius 10 is heated so that the temperature at any point that makes an angle of θ with the north pole has temperature $3\cos^2\theta + 2$. Find the resulting steady state temperature throughout the sphere. [*Suggestion:* Note that $3x^2 + 2 = 3P_0(x) + 2P_2(x)$, where $P_n(x)$ is the nth Legendre polynomial].

5. Solve the PDE boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + t\frac{\partial^4 u}{\partial x^4} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \frac{\sin x}{x}. \end{aligned}$$