Math 3357 Spring 2015

Partial Differential Equations

Final Exam

Saturday, May 9

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ________________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Points	10	10	15	20	15	15	20	15
Score								

Total:______________________

1. Let $0 < a < \pi$ and let f be the 2π-periodic function which satisfies

$$f(x) = \begin{cases} \dfrac{1}{2a} & \text{if } |x| \le a, \\ \\ 0 & \text{if } a < |x| < \pi. \end{cases}$$

a. Find the Fourier series for f.

b. Use your answer to part **a** to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(na)\cos(na)}{n}$.

2. Assume that $f \in L^1(\mathbb{R})$ and that $a \in \mathbb{R}$.

a. Show that

$$\mathcal{F}\left(e^{iax} f(x)\right)(\omega) = \mathcal{F}\left(f(x)\right)(\omega - a),$$

where $\mathcal{F}$ denotes the Fourier transform.

b. Show that

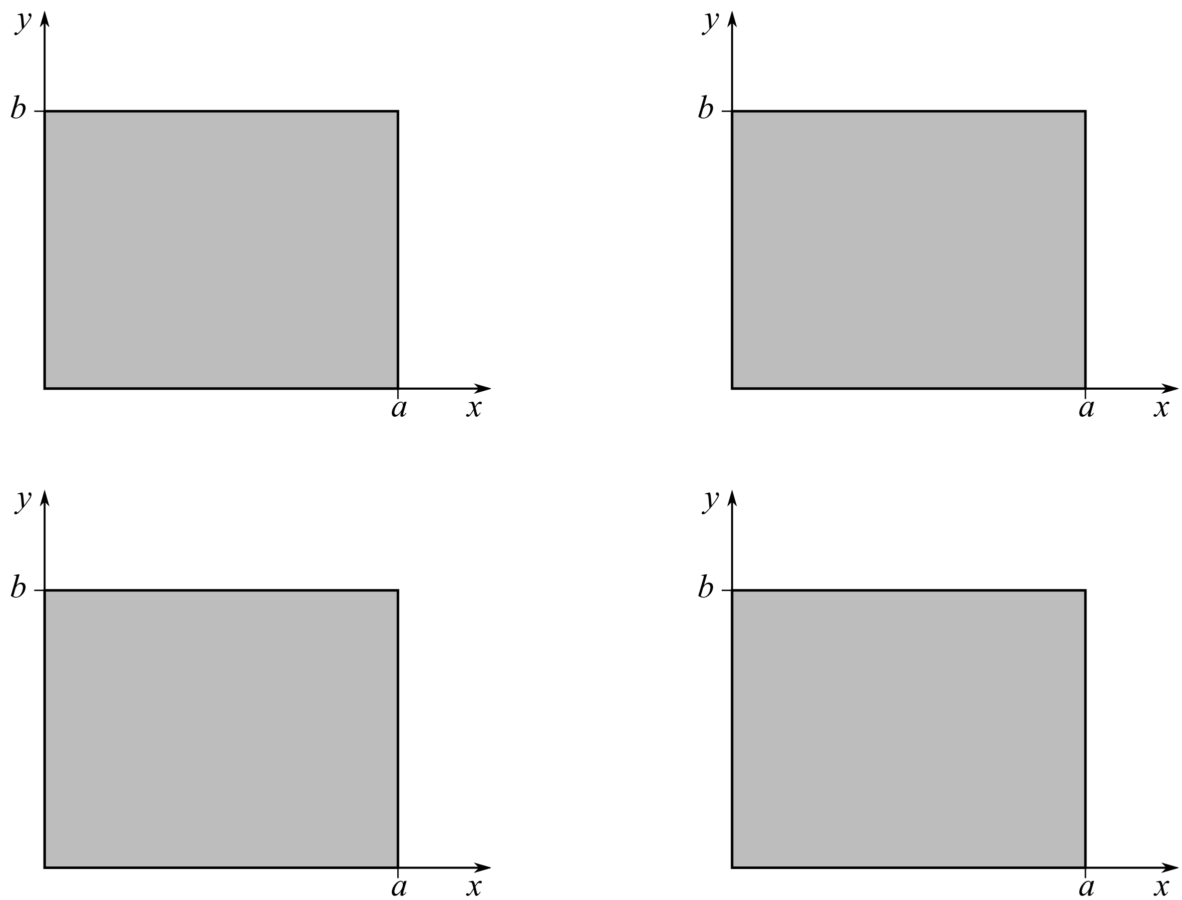
$$\mathcal{F}\left(\cos(ax) f(x)\right)(\omega) = \frac{\widehat{f}(\omega - a) + \widehat{f}(\omega + a)}{2},$$

where $\widehat{f} = \mathcal{F}\left(f(x)\right)$. [*Suggestion:* Use part **a**.]

3. Consider the PDE boundary value problem

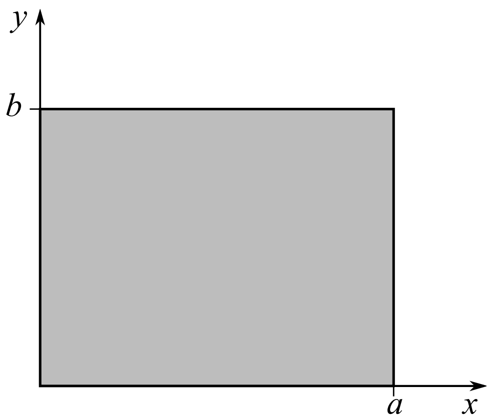
$$
\begin{aligned}
&\Delta u = 0, \;\; 0 < x < a, \;\; 0 < y < b,\\
&u(x,0) = f_1(x), \;\; 0 < x < a,\\
&u(x,b) = f_2(x), \;\; 0 < x < a,\\
&u_x(0,y) = g_1(y), \;\; 0 < y < b,\\
&u(a,y) - 2u_x(a,y) = g_2(y), \;\; 0 < y < b.
\end{aligned}
$$

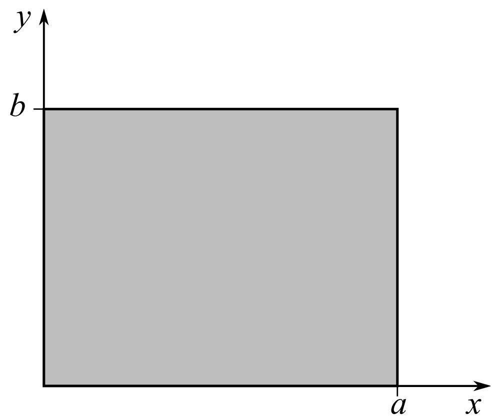
a. Decompose this problem into four subproblems, each of which may be solved using separation of variables. Define these problems by labelling the interiors and boundaries of the diagrams below.



b. How are the solutions u_1, u_2, u_3 and u_4 of the problems in part **a** related to the solution u of the original problem? What fact are you using that guarantees this?

c. Suppose we replace the given PDE with the wave equation $u_{tt} = c^2 \Delta u$. If $v(x, y)$ is the steady state solution, and $w(x, y, t) = u(x, y, t) - v(x, y)$, what are the PDE boundary value problems that v and w solve? Define these problems by labelling the interiors and boundaries of the diagrams below.





4. Consider the ODE

$$2x^2y'' + (x + x^2)y' - y = 0, \quad x > 0. \tag{1}$$

a. Show that $x = 0$ is a regular singular point of (1).

b. Determine the two values of r for which (1) may have a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$ ($a_0 \neq 0$). Why are we guaranteed that *both* values of r will yield solutions?

c. Find the recurrence relation satisfied by the coefficients a_n defined in part **b**.

d. Take $a_0 = 1$ (for each value of r) in part **b** and determine the next 3 coefficients.

4. (continued)

5. Find the eigenvalues and eigenfunctions of the ODE boundary value problem

$$\begin{aligned} &y'' + \lambda y = 0, \quad 0 < x < L, \\ &y(0) = y'(L) = 0. \end{aligned}$$

[*Suggestion:* Begin by showing that there are no eigenfunctions for $\lambda \leq 0$.]

6. Solve the boundary value problem

$$3y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2u, \quad u(x,0) = f(x), \quad -\infty < x < \infty, \quad y > 0$$

using either the method of characteristics or the Fourier transform.

7. Consider Laplace's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

on the region $0 < r < a$, $0 < \theta < \pi/2$ with the boundary conditions

$$\begin{aligned}
&u(r, 0) = 0, \;\; 0 < r < a, \\
&u(r, \pi/2) = 0, \;\; 0 < r < a, \\
&u(a, \theta) = \sin(2\theta), \;\; 0 < \theta < \pi/2, \\
&|u(r, \theta)| \text{ bounded as } r \to 0^+.
\end{aligned}$$

a. Provide a physical interpretation of this problem.

b. Solve the problem using separation of variables. [*Recall:* The solutions of the Euler equation $x^2y'' + axy' + by = 0$ are $y = c_1x^{r_1} + c_2x^{r_2}$, where $r_1 \neq r_2$ are the roots of the indicial equation $r^2 + (a-1)r + b = 0$.]

7. (continued)

8. The motion of a chain of length L suspended from a fixed point, subject to a constant gravitational acceleration g, can be modeled by the PDE boundary value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = g\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right), \quad 0 < x < L, \quad t > 0, \\ &u(L,t) = 0, \quad t > 0. \end{aligned}$$

Using separation of variables one can show that the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right)\left(A_n \cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_n}{2}t\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_n}{2}t\right)\right), \tag{2}$$

where α_n is the nth positive zero of J_0.

a. Recall that the parametric Bessel functions $J_0(\alpha_n r/a)$ are pairwise orthogonal on the interval $[0,a]$ relative to the weight function $w(r) = r$. Use this fact to show that the functions

$$X_n(x) = J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right)$$

are pairwise orthogonal on the interval $[0,L]$ relative to the weight function $w(x) = 1$. [*Suggestion:* Perform the substitution $r = \sqrt{x}$.]

b. Use the complementary relation $\displaystyle\int_0^a J_0^2(\alpha_n r/a) r\, dr = a^2 J_1^2(\alpha_n)/2$ to show that the inner product of X_n with itself on the interval $[0,L]$ is equal to $LJ_1^2(\alpha_n)$.

c. Use parts **a** and **b** to express the coefficients A_n and B_n of the general solution (2) in terms of the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = v(x)$.

8. (continued)