Math 3357 Spring 2014

Partial Differential Equations

Final Exam

Friday, May 9

Your name (please print):

Instructions: This is an open book, open notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**
Points	15	20	35	15	15	30
Score						

Total:____________________

1. Use the method of characteristics to find the solution to

$$\begin{aligned} &\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 3u, \\ &u(0, y) = e^y. \end{aligned}$$

2. Let

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ x - 1 & \text{if } 1 \leq x < 2. \end{cases}$$

a. Find the sine series expansion of f.

b. Find the cosine series expansion of f.

c. Without performing any additional integral computations, solve the heat problem

$$\begin{aligned}
&u_t = c^2 u_{xx}, \quad 0 < x < 2, \quad t > 0,\\
&u(0,t) = u(2,t) = 0,\\
&u(x,0) = f(x).
\end{aligned}$$

3. Consider the boundary value problem

$$\nabla^2 u = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{1}$$

$$u_y(x,0) = u_y(x,b) = 0, \quad 0 < x < a, \tag{2}$$

$$u(0,y) = u_x(0,y), \quad 0 < y < b, \tag{3}$$

$$u(a,y) = f(y), \quad 0 < y < b. \tag{4}$$

a. Show that separation of variables in (1) yields the ODE system

$$X'' - kX = 0, \quad 0 < x < a, \tag{5}$$

$$Y'' + kY = 0, \quad 0 < y < b. \tag{6}$$

What are the boundary conditions on Y that result from (2)? What is the boundary condition on X that results from (3)?

b. Determine the nontrivial solutions to (6) (with the boundary conditions from **a**) and the corresponding values of k. [*Remark: Do not* simply assume that $k > 0$.]

c. For each value of k found in part **b**, solve (5) (with the boundary conditions from **a**).

d. Use the principle of superposition to find the general series solution to (1)–(4). Be sure to give formulae (in terms of f) for the coefficients in your solution.

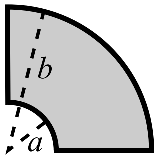
4. The temperature along one quarter of the circumference of a thin circular plate with unit radius is held at 100°, while the temperature along the rest of the circumference is kept at 0°. Assuming that heat can only move laterally through the plate, find an expression for the steady state temperature eventually achieved throughout the plate.

5. Use the method of Fourier transforms to solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} \pi & \text{if } |x| < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

6. Consider an ideal quarter-annular membrane with fixed edges, as shown below. If the membrane is set in motion, its displacement $u(r,\theta,t)$ from equilibrium at (polar) position (r,θ) and time t is governed by the PDE boundary value problem



$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad a < r < b, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{7}$$

$$u(r,0,t) = u(r,\pi/2,t) = u(a,\theta,t) = u(b,\theta,t) = 0. \tag{8}$$

a. Show that separation of variables in (7) leads to the ODE system

$$\Theta'' + \ell\Theta = 0, \quad 0 < \theta < \frac{\pi}{2}, \tag{9}$$

$$r^2R'' + rR' + (-kr^2 - \ell)R = 0, \quad a < r < b, \tag{10}$$

$$T'' - c^2kT = 0, \quad t > 0. \tag{11}$$

What are the boundary conditions (if any) on Θ, R and T that result from (8)?

b. Determine the solution to the eigenvalue problem (9) (with the boundary conditions given in **a**). You do not need to justify your answer.

c. Put (10) (with the boundary conditions given in **a**) in Sturm-Liouville form, and carefully explain why it is a *regular* problem. [*Remark:* The separation constant ℓ should be viewed as a fixed constant.]

d. Part **c** guarantees that for each value of ℓ, the eigenvalues of (10) form an unbounded increasing sequence:

$$-k_{\ell,1} < -k_{\ell,2} < -k_{\ell,3} < \cdots,$$

and that the corresponding eigenfunctions are unique (up to scalar multiples):

$$R_{\ell,1}, R_{\ell,2}, R_{\ell,3}, \ldots.$$

One can also show that the eigenvalues are, in fact, all positive: $-k_{\ell,m} = \lambda_{\ell,m}^2 > 0$ (*do not* attempt to verify any of these statements). Use this information to solve (11) (in terms of the $\lambda_{\ell,m}$).

e. Use the principle of superposition to write down the (double) series form of the general solution to this vibrating membrane problem. [*Remark: Do not* attempt to explicitly determine the eigenfunctions $R_{\ell,m}$, and *do not* give integral formulae for the coefficients.]