Exercise 1. Verify that both $u = \log(x^2+y^2)$ and $u = \arctan(y/x)$ are solutions of Laplace's equation $u_{xx} + u_{yy} = 0$.

Exercise 2. Solve the boundary value problem.

a. $$r\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = e^{3x}, \quad (x, y) \in \mathbb{R} \times (0, \infty),$$

$$u(x, 0) = f(x)$$

b. $$\frac{\partial u}{\partial x} - 3y\frac{\partial u}{\partial y} = 0, \;\; (x, y) \in (0, \infty) \times \mathbb{R},$$

$$u(0, y) = y^4 - 2$$

c. $$\frac{\partial u}{\partial x} - 2u\frac{\partial u}{\partial y} = 0, \;\; (x, y) \in (0, \infty) \times \mathbb{R},$$

$$u(0, y) = y$$

d. $$4x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2y, \;\; (x, y) \in \mathbb{R} \times (0, \infty),$$

$$u(x, 0) = \log(8 + x^2)$$

Exercise 3. Show that the general solution to $u_{xy} + u_x = 0$ has the form $u(x, y) = F(y) + e^{-y}G(x)$. [*Suggestion:* Notice that $u_{xy} + u_x = (u_y + u)_x$.]

Exercise 4. Solve the wave equation subject to the initial conditions

$$u(x, 0) = xe^{-x^2}, \;\; u_t(x, 0) = \frac{1}{1 + x^2}, \;\; x \in \mathbb{R}.$$

Exercise 5. Suppose we want to find a solution of the (unbounded) wave equation that consists of a single traveling wave moving to the right with shape given by the graph of $f(x)$. What initial conditions are required to cause this to happen?

Exercise 6. This problem concerns the partial differential equation

$$u_{xx} + 4u_{xy} + 3u_{yy} = 0. \tag{1}$$

a. If F and G are twice differentiable functions, show that

$$u(x,y) = F(3x - y) + G(x - y) \tag{2}$$

is a solution to (1).

b. Use a linear change of variables to show that every solution to (1) has the form (2).

c. Find the solution to (1) that satisfies the initial conditions

$$u(x,0) = \frac{x}{x^2+1} \text{ and } u_y(x,0) = 0 \text{ for all } x.$$

Exercise 7. Show that the functions

$$\cos x, \cos 3x, \cos 5x, \cos 7x, \ldots,$$

are pairwise orthogononal relative to the inner product $\langle f, g\rangle = \int_0^{\pi/2} f(x)g(x)\,dx$. [*Suggestion:* Use the identity $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$.]

Exercise 8. Let

$$\begin{aligned} f_1(x) &= 1, \\ f_2(x) &= 2x - 1, \\ f_3(x) &= 6x^2 - 6x + 1, \\ f_4(x) &= 20x^3 - 30x^2 + 12x - 1. \end{aligned}$$

a. Verify that the polynomials f_1, f_2, f_3 and f_4 are pairwise orthogonal relative to the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

b. Let $p(x) = x^3 - 2$. Use part **a** to write p as a linear combination of f_1, f_2, f_3 and f_4. [*Suggestion:* Recall that since the f_i are orthogonal, if

$$p = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4,$$

then $a_j = \langle p, f_j\rangle / \langle f_j, f_j\rangle$.]

c. Explain why the procedure of part **b** *fails* if we take $p(x) = x^5 - 2x + 1$.

Exercise 9. For each 2π-periodic function f given: **i.** carefully sketch three periods of f and **ii.** carefully sketch three periods of the the Fourier series of f.

a. $f(x) = \begin{cases} \lfloor 2x/\pi \rfloor & \text{if } -\pi \le x < \pi, \\ f(x+2\pi) & \text{otherwise.} \end{cases}$

b. $f(x) = \begin{cases} \min\{|3x/\pi|, 1\} & \text{if } -\pi \le x < \pi, \\ f(x+2\pi) & \text{otherwise.} \end{cases}$

c. $f(x) = \begin{cases} \pi/2 & \text{if } -\pi \le x < 0, \\ \pi - x/2 & \text{if } 0 \le x < \pi, \\ f(x+2\pi) & \text{otherwise.} \end{cases}$