Partial Differential Equations — Exam 2 Review Exercises
Spring 2018

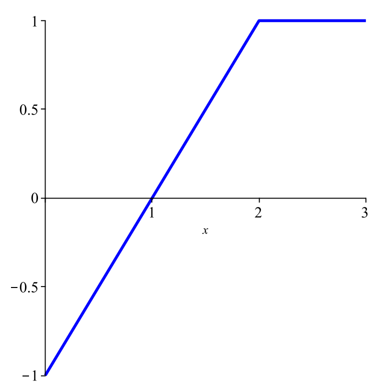
Exercise 1. Carefully state the Fourier Series Representation Theorem (for functions of arbitrary period).

Exercise 2. Find the Fourier series for the $2p$-periodic function given on the interval $[-p, p)$ by

$$f(x) = \begin{cases} x + p & \text{if } -p \le x < 0, \\ p & \text{if } 0 \le x < p. \end{cases}$$

Sketch the graph of f and the graph of its Fourier series for 3 periods.

Exercise 3. Sketch the even and odd 6-periodic extensions (for at least 3 periods) of the function whose graph is shown below.



Exercise 4. Find the sine and cosine series expansions of the function $g(x) = 2 + x - x^2$, $0 < x < 2$.

Exercise 5. Consider the heat boundary value problem

$$\begin{array}{ll} u_t = c^2 u_{xx}, & t > 0,\ 0 < x < L, \\ u_x(0,t) = 0, & t > 0, \\ u_x(L,t) = -\kappa u(L,t), & t > 0, \\ u(x,0) = f(x), & 0 < x < L \end{array}$$

in which κ is a positive constant.

a. Provide a physical interpretation of this problem.

b. Use separation of variables and superposition to find the solution to this problem.

Exercise 6. Use the series solution $u(x,t)$ of the fixed endpoint vibrating string problem to show that

$$u(x, t + L/c) = -u(L - x, t).$$

What does this imply about the shape of the string at half a time period?

Exercise 7. A ideal elastic string of length 4 moves according to the PDE $u_{tt} = 4u_{xx}$. It is initially deformed into the shape of the graph of the function

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ x - 1 & \text{if } 1 < x < 2, \\ 3 - x & \text{if } 2 < x < 3, \\ 0 & \text{if } 3 < x < 4, \end{cases}$$

and given a uniform unit downward speed. Determine its position at any later time.

Exercise 8. Obtain the Fourier series for the $2p$-periodic function given by $f(x) = x(2p-x)$ for $0 \leq x \leq 2p$ by translating the Fourier series of Textbook Exercise 2.3.3.

Exercise 9. [Extra Credit] Obtain the Fourier series of Textbook Exercise 2.2.20a by multiplying $h(x) = \sin x$ by an appropriate square wave and simplifying. [Suggested by Regis Noubiap.]