Exercise 1. Carefully state the Fourier Series Representation Theorem (for functions of arbitrary period).

Theorem. Let f be a $2p$-periodic piecewise smooth function. Then

$$\frac{f(x+)+f(x-)}{2} = a_0 + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)\,dx, \;\; a_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\cos\left(\frac{n\pi x}{p}\right)\,dx, \;\; b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Exercise 2. Find the Fourier series for the $2p$-periodic function given on the interval $[-p,p)$ by

$$f(x) = \begin{cases} x+p & \text{if } -p \le x < 0, \\ p & \text{if } 0 \le x < p. \end{cases}$$

Sketch the graph of f and the graph of its Fourier series for 3 periods.

Solution. Because the function in question lacks odd/even symmetry, we need to compute all of its Fourier coefficients manually. We have

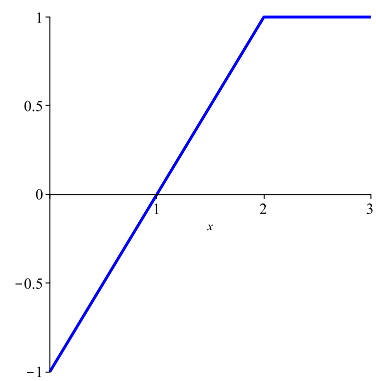
$$\begin{aligned}
a_0 &= \frac{1}{2p}\int_{-p}^{p} f(x)\,dx = \frac{1}{2p}\left(\int_{-p}^{0}(x+p)\,dx + \int_0^p p\,dx\right) \\
&= \frac{1}{2p}\left(\frac{x^2}{2} + px\Big|_{-p}^{0} + p^2\right) = \frac{3p}{4}. \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\,\cos\left(\frac{n\pi x}{p}\right)\,dx = \frac{1}{p}\left(\int_{-p}^{0}(x+p)\cos\left(\frac{n\pi x}{p}\right)\,dx + \int_0^p p\cos\left(\frac{n\pi x}{p}\right)\,dx\right) \\
&= \int_{-p}^{p}\cos\left(\frac{n\pi x}{p}\right)\,dx + \frac{1}{p}\int_{-p}^{0} x\cos\left(\frac{n\pi x}{p}\right)\,dx \\
&= \frac{p}{n\pi}\sin\left(\frac{n\pi x}{p}\right)\Big|_{-p}^{p} + \frac{1}{p}\left(\frac{px}{n\pi}\sin\left(\frac{n\pi x}{p}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right)\right)\Big|_{-p}^{0} \\
&= \frac{p}{n^2\pi^2}(1-\cos(n\pi)) = \frac{p}{n^2\pi^2}(1-(-1)^n).
\end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\left(\frac{n\pi x}{p}\right)\,dx = \frac{1}{p}\left(\int_{-p}^{0}(x+p)\sin\left(\frac{n\pi x}{p}\right)\,dx + \int_0^p p\sin\left(\frac{n\pi x}{p}\right)\,dx\right)\\
&= \int_{-p}^{p}\sin\left(\frac{n\pi x}{p}\right)\,dx + \frac{1}{p}\int_{-p}^{0} x\sin\left(\frac{n\pi x}{p}\right)\,dx\\
&= -\frac{p}{n\pi}\cos\left(\frac{n\pi x}{p}\right)\Bigg|_{-p}^{p} + \frac{1}{p}\left(\frac{px}{n\pi}\cos\left(\frac{n\pi x}{p}\right) - \frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right)\Bigg|_{-p}^{0}\\
&= \frac{(-1)^{n+1}p}{n\pi}
\end{aligned}
$$

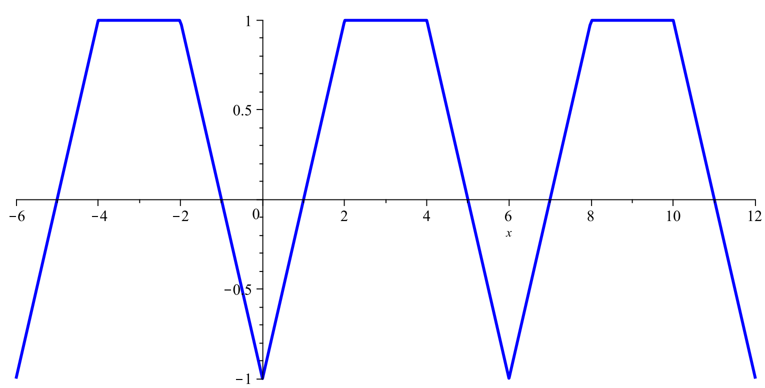
The Fourier series is therefore

$$
\boxed{\frac{3p}{4} + \sum_{n=1}^{\infty}\frac{p}{n^2\pi^2}(1-(-1)^n)\cos\left(\frac{n\pi x}{p}\right) + \frac{(-1)^{n+1}p}{n\pi}\sin\left(\frac{n\pi x}{p}\right).}
$$

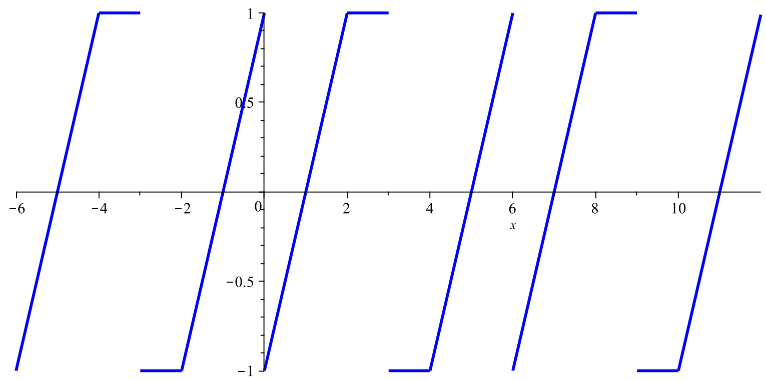
Exercise 3. Sketch the even and odd 6-periodic extensions (for at least 3 periods) of the function whose graph is shown below.



The even extension:



The odd extension:



Exercise 4. Find the sine and cosine series expansions of the function $g(x) = 2 + x - x^2$, $0 < x < 2$.

Solution. Once again, there's no particularly nice symmetry so we just hammer out the coefficients using Euler's Formulae. For the cosine expansion we have

$$a_0 = \frac{1}{2}\int_0^2 2 + x - x^2\, dx = \frac{1}{2}\left(2x + \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^2\right) = \frac{5}{3}$$

and for $n \geq 1$

$$\begin{aligned} a_n &= \frac{2}{2}\int_0^2 (2 + x - x^2)\cos\left(\frac{n\pi x}{2}\right)\, dx \\ &= (2 + x - x^2)\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + (1 - 2x)\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{16}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)\bigg|_0^2 \\ &= \frac{1}{n^2\pi^2}\left(-12\cos(n\pi) - 4\right) = \frac{12(-1)^{n+1} - 4}{n^2\pi^2} \end{aligned}$$

so that the cosine series for g is

$$\boxed{\frac{5}{3} + \sum_{n=1}^{\infty} \frac{12(-1)^{n+1} - 4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right).}$$

The sine series coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^2 (2 + x - x^2)\sin\left(\frac{n\pi x}{2}\right)\, dx \\ &= -(2 + x - x^2)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + (1 - 2x)\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right) - \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi x}{2}\right)\bigg|_0^2 \\ &= -\frac{16}{n^3\pi^3}\cos(n\pi) + \frac{4}{n\pi} + \frac{16}{n^3\pi^3} = \frac{16}{n^3\pi^3}((-1)^{n+1} + 1) + \frac{4}{n\pi} \end{aligned}$$

which gives the sine expansion

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{16}{n^3\pi^3}((-1)^{n+1} + 1) + \frac{4}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right).}$$

Exercise 5. Consider the heat boundary value problem

$$\begin{array}{ll} u_t = c^2 u_{xx}, & t > 0,\ 0 < x < L, \\ u_x(0,t) = 0, & t > 0, \\ u_x(L,t) = -\kappa u(L,t), & t > 0, \\ u(x,0) = f(x), & 0 < x < L \end{array}$$

in which κ is a positive constant.

a. Provide a physical interpretation of this problem.

Solution. The function u models the temperature in a perfectly (laterally) insulated heat conductive rod of length L. It's left end is also perfectly insulated, while its right end radiates thermal energy at a rate proportional to its temperature there. The function $f(x)$ describes the *initial* temperature throughout the rod.

b. Use separation of variables and superposition to find the solution to this problem.

Solution. We first seek separated solutions of the form $u(x,t) = X(x)T(t)$ satisfying all of the homogeneous linear requirements, i.e. the first three conditions. Substituting the separated solution into the PDE yields

$$XT' = c^2 X'' T \;\Rightarrow\; \frac{X''}{X} = \frac{T'}{c^2 T} = k \text{ (constant)}$$

since the two sides of the latter equation are functions of distinct independent variables. This gives us the pair of separated ODEs

$$X'' - kX = 0,\;\; T' - kc^2 T = 0.$$

From the first boundary condition we obtain

$$X'(0)T(t) = 0 \;\Rightarrow\; X'(0) = 0$$

since we do not want $T \equiv 0$. The second boundary condition requires that

$$X'(L)T(t) = -\kappa X(L)T(t) \;\Rightarrow\; X'(L) = -\kappa X(L).$$

We have thus obtained an ODE boundary value problem in X that requires a case by case analysis of the possible values of k for which there are nontrivial (nonzero) solutions.

Case 1: $k = \mu^2 > 0$. In this situation the ODE for X becomes $X'' - \mu^2 X = 0$ with characteristic equation $r^2 - \mu^2 = 0$, whose roots are $r = \pm\mu$. The solutions are then given by

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The boundary conditions require that

$$\begin{array}{l} \mu\left(c_1 e^{\mu\cdot 0} - c_2 e^{-\mu\cdot 0}\right) = 0 \;\Rightarrow\; c_1 - c_2 = 0, \\ \mu\left(c_1 e^{\mu L} - c_2 e^{-\mu L}\right) = -\kappa(c_1 e^{\mu L} + c_2 e^{-\mu L}) \;\Rightarrow\; c_1(\kappa+\mu)e^{\mu L} + c_2(\kappa-\mu)e^{-\mu L} = 0 \end{array}$$

or in matrix form

$$\begin{pmatrix} 1 & -1 \\ (\kappa+\mu)e^{\mu L} & (\kappa-\mu)e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$(\kappa-\mu)e^{-\mu L}+(\kappa+\mu)e^{\mu L}=\kappa(e^{\mu L}+e^{-\mu L})+\mu(e^{\mu L}-e^{-\mu L})>0,$$

which means that the only possibility is that $c_1=c_2=0$, or in other words $X\equiv 0$. So we must move on to the next case.

Case 2: $k=0$. Now the ODE in X simplifies to $X''=0$ which means that $X=ax+b$. The first boundary condition immediately implies $a=0$ and the second then becomes

$$0=-\kappa b \;\Rightarrow\; b=0$$

which once again tells us that $X\equiv 0$. So we move on.

Case 2: $k=-\mu^2<0$. Things finally get interesting. The ODE becomes $X''+\mu^2X=0$ whose characteristic equation has roots $\pm i\mu$, so that $X=c_1\cos\mu x+c_2\sin\mu x$. From the first boundary condition we find

$$-c_1\sin 0+c_2\cos 0=0 \;\Rightarrow\; c_2=0.$$

The second boundary condition is then

$$-\mu c_1\sin\mu L=-\kappa c_1\cos\mu L \;\Rightarrow\; \tan\mu L=\frac{\kappa}{\mu},$$

since we *do not* want $c_1=0$ at this point. The diagram below illustrates that this equation has an increasing sequence of positive solutions, which we label

$$0<\mu_1<\mu_2<\mu_3<\cdots$$

Up to the choice of the scalar, these finally give the nontrivial solutions

$$X_n=\cos\mu_n x,\;\; n\in\mathbb{N}.$$

Since $-\mu^2=k$, for each $n\in N$ the ODE for T becomes

$$T'+(c\mu_n)^2T=0 \;\Rightarrow\; T'=-(c\mu_n)^2T \;\Rightarrow\; T=T_n=c_ne^{-(c\mu_n)^2t}.$$

We have finally obtained our separated solutions:

$$u_n(x,t)=c_ne^{-(c\mu_n)^2t}\cos\mu_n x,\;\; n\in\mathbb{N},$$

where μ_n is the nth positive solution to $\tan\mu L=\kappa/\mu$.

Superposition: By the principle of superposition, the function

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}c_ne^{-(c\mu_n)^2t}\cos\mu_n x}$$

is guaranteed to satisfy the homogeneous linear restrictions we have imposed on the individual summands. Put another way, we've solved the given problem, up to the initial condition, which we now attempt to impose:

$$f(x)=u(x,0)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}c_ne^{-(c\mu_n)^2\cdot 0}\cos\mu_n x=\sum_{n=1}^{\infty}c_n\cos\mu_n x. \tag{1}$$

One can easily check that the functions $\cos \mu_n x$ are pairwise orthogonal on the interval $[0, L]$. Moreover, if f is "sufficiently smooth" one can be assured that an expression of f as a linear combination of these functions of the form (1) is always possible. This is a consequence of what is known as "Sturm-Liouville Theory," a subject we may touch upon later.

In any case, this tells us how to "extract" the coefficients c_n of the solution to our original problem from the intial data $f(x)$: as ratios of inner products. Specifically

$$\boxed{c_n = \frac{\langle f(x), \cos \mu_n x \rangle}{\langle \cos \mu_n x, \cos \mu_n x \rangle} = \frac{\int_0^L f(x) \cos \mu_n x \, dx}{\int_0^L \cos^2 \mu_n x \, dx}.}$$

And, up to the evaluation of the integrals, this furnishes the solution.

Exercise 6. Use the series solution $u(x,t)$ of the fixed endpoint vibrating string problem to show that

$$u(x, t + L/c) = -u(L - x, t).$$

What does this imply about the shape of the string at half a time period?

Solution. This is, more or less, simply an exercise in applying the addition formulae for the sine and cosine functions. The general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\mu_n x) \left(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t) \right) \tag{2}$$

where $\mu_n = n\pi/L$ and $\lambda_n = c\mu_n$. Rather than drag the entire series through our investigation, let's simply look at the relevant terms. Notice that

$$\begin{aligned} \cos(\lambda_n(t + L/c)) &= \cos(\lambda_n t + n\pi) = \cos(\lambda_n t)\cos(n\pi) - \sin(\lambda_n t)\sin(n\pi) = (-1)^n \cos(\lambda_n t), \\ \sin(\lambda_n(t + L/c)) &= \sin(\lambda_n t + n\pi) = \sin(\lambda_n t)\cos(n\pi) + \cos(\lambda_n t)\sin(n\pi) = (-1)^n \sin(\lambda_n t). \end{aligned}$$

Furthermore

$$\sin(\mu_n(L - x)) = \sin(n\pi - \mu_n x) = \sin(n\pi)\cos(\mu_n x) - \sin(\mu_n x)\cos(n\pi) = (-1)^{n+1}\sin(\mu_n x)$$

Substituting these results into the solution (2), the conclusion follows immediately.

Exercise 7. A ideal elastic string of length 4 moves according to the PDE $u_{tt} = 4u_{xx}$. It is initially deformed into the shape of the graph of the function

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ x - 1 & \text{if } 1 < x < 2, \\ 3 - x & \text{if } 2 < x < 3, \\ 0 & \text{if } 3 < x < 4, \end{cases}$$

and given a uniform unit downward speed. Determine its position at any later time.

Solution. As stated in the previous exercise, we know the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\mu_n x)\left(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)\right)$$

where the b_n are the sine series coefficients of $f(x)$ and the b_n^* are "almost" the sine series coefficients of the initial velocity function, which in this case is $g(x) = -1$.

We could play various games with existing Fourier series to find the sine expansion of $f(x)$, but to keep things simple we'll just defer to Euler's Formula:

$$\begin{aligned}
b_n &= \frac{2}{4}\int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\left(\int_1^2 (x-1)\sin\left(\frac{n\pi x}{4}\right)dx + \int_2^3 (3-x)\sin\left(\frac{n\pi x}{4}\right)dx\right) \\
&= \frac{1}{2}\left(-(x-1)\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right) + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi x}{4}\right)\right)\Bigg|_1^2 \\
&\quad + \frac{1}{2}\left(-(3-x)\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi x}{4}\right)\right)\Bigg|_2^3 \\
&= \frac{16\sin\left(\frac{n\pi}{2}\right) - 8\sin\left(\frac{n\pi}{4}\right) - 8\sin\left(\frac{3n\pi}{4}\right)}{n^2\pi^2}
\end{aligned}$$

Fortunately the b_n^* coefficients are much easier to obtain. The sine expansion of $g(x) = -1$ on the interval $0 < x < 4$ is simply the negation of Exercise 2.3.1, with $p = 4$. That is,

$$-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi x}{4}\right)$$

Therefore we only need the odd indexed modes and have

$$b_{2k+1}^* = \frac{1}{\lambda_{2k+1}}\left(-\frac{4}{\pi(2k+1)}\right) = \frac{L}{c(2k+1)\pi}\left(\frac{-4}{\pi(2k+1)}\right) = \frac{-2L}{\pi^2(2k+1)^2}.$$

We conclude that the shape of the string at any time is given by

$$\begin{aligned}
u(x,t) = &\sum_{n=1}^{\infty}\left(\frac{16\sin\left(\frac{n\pi}{2}\right) - 8\sin\left(\frac{n\pi}{4}\right) - 8\sin\left(\frac{3n\pi}{4}\right)}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{n\pi t}{2}\right) \\
&+ \sum_{k=0}^{\infty}\frac{-2L}{\pi^2(2k+1)^2}\sin\left(\frac{(2k+1)\pi x}{4}\right)\cos\left(\frac{(2k+1)\pi x}{2}\right).
\end{aligned}$$

Exercise 8. Obtain the Fourier series for the $2p$-periodic function given by $f(x) = x(2p-x)$ for $0 \le x \le 2p$ by translating the Fourier series of Textbook Exercise 2.3.3.

Solution. The function $f(x)$, on the interval $[0, 2p]$, is a downward opening parabola with roots at $x = 0$ and $x = 2p$. Its maximum value on this interval occurs in the center, when $x = p$, which is $f(p) = p^2$. This is precisely the function of Exercise 2.3.3 with $a = p^2$, shifted to the right by p units. Using the Fourier series given there we therefore obtain the translated series

$$
\begin{aligned}
&\frac{2p^2}{3}+4p^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2}\cos\left(\frac{n\pi(x-p)}{p}\right)\\
&\qquad=\frac{2p^2}{3}+4p^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2}\left(\cos\left(\frac{n\pi x}{p}\right)\cos\left(n\pi\right)+\sin\left(\frac{n\pi x}{p}\right)\sin\left(n\pi\right)\right)\\
&\qquad=\frac{2p^2}{3}+4p^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right)(-1)^n\\
&\qquad=\boxed{\frac{2p^2}{3}-4p^2\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right).}
\end{aligned}
$$

Exercise 9. [Extra Credit] Obtain the Fourier series of Textbook Exercise 2.2.20a by multiplying $h(x) = \sin x$ by an appropriate square wave and simplifying. [Suggested by Regis Noubiap.]