Exercise 1. An ideal elastic membrane with dimensions 1×1 and $c = 2$ is initially deformed into the shape of the graph of the function $f(x, y) = 100(1 - x - y)x(1 - x)y(1 - y)$ and imparted with a velocity given at each point by $g(x, y) = x - y$. Determine the shape of the membrane at any later time t.

Exercise 2. A thin rectangular $a \times b$ metal plate with insulated faces has one pair of opposite edges held constantly at $0°$ while the other pair of opposite edges is insulated. If the plate is initially heated so as to have temperature $f(x, y)$ at each point throughout its interior, determine the temperature at any later time.

Exercise 3. Solve the Dirichlet problem on the interior of a 2×1 rectangle subject to the boundary conditions $u(x, 0) = x^2$, $u(2, y) = 4 - y$, $u(x, 1) = 5 - x$, $u(0, y) = 5y$.

Exercise 4. Show that the function

$$f(x, y) = \frac{x^3 - 3xy^2}{(x^2 + y^2)^3}$$

is harmonic.

Exercise 5. A thin circular metal disk of radius 3 with insulated faces has the temperature along its edge held at $0°$ in the first quadrant, $50°$ in the second quadrant, $0°$ in the third quadrant and $25°$ in the fourth quadrant. Find the resulting steady-state temperature distribution throughout the disk.

Exercise 6. Show that

$$f(x) = \frac{x}{x^2 + x - 2}$$

is analytic at $a = 0$. [*Suggestion:* First find the partial fraction decomposition of f.]

Exercise 7. Show that $a = 0$ is an ordinary point of the second order ODE

$$(2 + x^2)y'' - xy' - 3y = 0. \tag{1}$$

Find the recursion relation satisfied by the coefficients of the power series expansion centered at $a = 0$ of any solution to (1), and give a lower bound on its radius of convergence. Find explicit expressions for the coefficients in each of two linearly independent solutions and compute their radii of convergence exactly.

Exercise 8. Find the two values of r for which

$$2x^2y'' + 3xy' + (2x^2 - 1)y = 0$$

has solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0,$$

in which the power series has positive radius of convergence. For each value of r find the recursion relation satisfied by the coefficients a_n and (assuming $a_0 = 1$) the first 5 terms in the series. If possible, find a general expression for a_n in each case.