Exercise 1. An ideal elastic membrane with dimensions 1×1 and $c = 2$ is initially deformed into the shape of the graph of the function $f(x, y) = 100(1 - x - y)x(1 - x)y(1 - y)$ and imparted with a velocity given at each point by $g(x, y) = x - y$. Determine the shape of the membrane at any later time t.

Solution. We need the double Fourier series coefficients for f and g. These are given by

$$\begin{aligned} B_{mn} &= \frac{4}{1 \cdot 1} \int_0^1 \int_0^1 100(1 - x - y)x(1 - x)y(1 - y) \sin(m\pi x) \sin(n\pi y)\, dy\, dx \\ &= 400 \left(\frac{12(1 - (-1)^{m+n})}{\pi^6 m^3 n^3} \right) = \frac{4800}{\pi^6} \left(\frac{(1 - (-1)^{m+n})}{m^3 n^3} \right), \\ B'_{mn} &= \frac{4}{1 \cdot 1} \int_0^1 \int_0^1 (x - y) \sin(m\pi x) \sin(n\pi y)\, dy\, dx \\ &= \frac{(-1)^n - (-1)^m}{\pi^2 mn}. \end{aligned}$$

Since $\lambda_{mn} = c\pi\sqrt{m^2 + n^2}$, we find that

$$B^*_{mn} = \frac{(-1)^n - (-1)^m}{\pi^3 cmn\sqrt{m^2 + n^2}}.$$

Hence the shape of the membrane is given by

$$\begin{aligned} u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg(&\frac{4800}{\pi^6} \left(\frac{(1 - (-1)^{m+n})}{m^3 n^3} \right) \cos\left(c\pi\sqrt{m^2 + n^2}\, t \right) \\ &+ \frac{(-1)^n - (-1)^m}{\pi^3 cmn\sqrt{m^2 + n^2}} \sin\left(c\pi\sqrt{m^2 + n^2}\, t \right) \Bigg) \sin(m\pi x) \sin(n\pi y). \end{aligned}$$

Exercise 2. A thin rectangular $a \times b$ metal plate with insulated faces has one pair of opposite edges held constantly at $0°$ while the other pair of opposite edges is insulated. If the plate is initially heated so as to have temperature $f(x, y)$ at each point throughout its interior, determine the temperature at any later time.

Solution. Let's assume the bottom and top edges are being held at $0°$, while the left and right edges are those that are insulated. This gives us the boundary value problem

$$\begin{aligned}
&u_t = c^2(u_{xx} + u_{yy}), \;\; 0 < x < a, \;\; 0 < y < b, \;\; t > 0,\\
&u(x,0,t) = u(x,b,t) = 0, \;\; 0 < x < a, \;\; t > 0,\\
&u_x(0,y,t) = u_x(a,y,t) = 0, \;\; 0 < y < b, \;\; t > 0,\\
&u(x,y,0) = f(x,y), \;\; 0 < x < a, \;\; 0 < y < b.
\end{aligned}$$

Since this is a boundary value problem we haven't encountered before, we need to separate variables, superimpose, and apply orthogonality.

We begin by assuming $u(x,y,t) = X(x)Y(y)T(t)$ and imposing all of the homogeneous conditions of the problem on u, i.e. all but the last. Plugging in to the heat equation we obtain

$$XYT' = c^2(X''YT + XY''T) \;\; \Rightarrow \;\; \frac{T'}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = A \;\; \text{(constant)},$$

since the two sides of the final equation are functions of distinct independent variables. Moving the Y term to the other side yields

$$\frac{X''}{X} = A - \frac{Y''}{Y} = B \;\; \text{(constant)}$$

by similar reasoning. If we let $C = A - B$ we find that we have the three separated ODEs

$$\begin{aligned}
T' &= c^2(B + C)T,\\
X'' - BX &= 0,\\
Y'' - CY &= 0.
\end{aligned}$$

The homogeneous boundary conditions on u imply that we must also have

$$\begin{aligned}
X'(0) &= X'(a) = 0,\\
Y(0) &= Y(b) = 0.
\end{aligned}$$

We previously solved both the X and Y boundary value problems. In the context of the fixed length vibrating string, we found that

$$Y = Y_n = \sin(\nu_n y), \;\; C = -\nu_n^2, \;\; \nu_n = \frac{n\pi}{b}, \;\; n \in \mathbb{N},$$

and in the one-dimensional insulated ends heat problem we found that

$$X = X_m = \cos(\mu_m x), \;\; B = -\mu_m^2, \;\; \mu_m = \frac{m\pi}{a}, \;\; m \in \mathbb{N}_0.$$

Using these values for B and C, the ODE for T becomes

$$T' = -c^2(\mu_m^2 + \nu_n^2)T \;\; \Rightarrow \;\; T = T_{mn} = c_{mn}e^{-c^2(\mu_m^2+\nu_n^2)t}.$$

We conclude that the normal modes for this particular heat problem are

$$u_{mn}(x,y,t) = X_m(x)Y_n(y)T_{mn}(t) = c_{mn}\cos(\mu_m x)\sin(\nu_n y)e^{-c^2(\mu_m^2+\nu_n^2)t}, \;\; (m,n) \in \mathbb{N}_0 \times \mathbb{N},$$

where μ_m, ν_n are as above.

Superposition gives the general solution

$$\boxed{u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\cos(\mu_m x)\sin(\nu_n y)e^{-c^2(\mu_m^2+\nu_n^2)t}}, \tag{1}$$

which still satisfies all of the homogeneous components of the boundary value problem. But because it is more flexible than the individual modes, we are in a position to finally impose the remaining initial condition:

$$f(x,y) = u(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\cos(\mu_m x)\sin(\nu_n y).$$

This is another type of double Fourier series. It is a linear combination of the functions $W_{mn}(x,y) = \cos(\mu_m x)\sin(\nu_n y)$ which one can readily show are orthogonal relative to the inner product

$$\langle g, h\rangle = \int_0^a\int_0^b g(x,y)h(x,y)\,dy\,dx.$$

Consequently we can extract the Fourier coefficients as ratios of inner products:

$$c_{mn} = \frac{\langle f, W_{mn}\rangle}{\langle W_{mn}, W_{mn}\rangle}.$$

Writing the inner products as integrals and evaluating the denominator one finds that

$$\boxed{c_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\cos(\mu_m x)\sin(\nu_n y)\,dy\,dx,\quad m\neq 0},$$

whereas

$$\boxed{c_{0n} = \frac{2}{ab}\int_0^a\int_0^b f(x,y)\sin(\nu_n y)\,dy\,dx}.$$

Together with (1), these give the complete solution to the problem.

Exercise 3. Solve the Dirichlet problem on the interior of a 2×1 rectangle subject to the boundary conditions $u(x,0) = x^2$, $u(2,y) = 4-y$, $u(x,1) = 5-x$, $u(0,y) = 5y$.

Solution. We need the half-range sine expansion of each edge condition. Unfortunately, with only one exception, these aren't expansions we've derived before. So we integrate:

$$\begin{aligned}
a_n &= \frac{2}{2}\int_0^2 x^2\sin\left(\frac{n\pi x}{2}\right)\,dx = \frac{8\left(\pi^2 n^2(-1)^{n+1} - 2 + 2(-1)^n\right)}{\pi^3 n^3},\\
b_n &= \frac{2}{2}\int_0^2 (5-x)\sin\left(\frac{n\pi x}{2}\right)\,dx = \frac{2\left(5+3(-1)^{n+1}\right)}{\pi n},\\
c_n &= \frac{2}{1}\int_0^1 5y\sin(n\pi y)\,dy = \frac{10(-1)^{n+1}}{\pi n},\\
d_n &= \frac{2}{1}\int_0^1 (4-y)\sin(n\pi y)\,dy = \frac{2\left(4+3(-1)^{n+1}\right)}{\pi n}.
\end{aligned}$$

Now recall that the coefficients in the solution to the Dirichlet problem are given by

$$A_n = \frac{a_n}{\sinh(n\pi/2)}, \quad B_n = \frac{b_n}{\sinh(n\pi/2)}, \quad C_n = \frac{c_n}{\sinh(2n\pi)}, \quad D_n = \frac{d_n}{\sinh(2n\pi)}$$

(here and below we are simply invoking the solution given on pp 167–168 of our textbook, with $a = 2$ and $b = 1$). It follows that the solution is

$$\begin{aligned} u(x,y) = &\sum_{n=1}^{\infty} \frac{8\left(\pi^2 n^2(-1)^{n+1} - 2 + 2(-1)^n\right)}{\pi^3 n^3 \sinh(n\pi/2)} \sin\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi(1-y)}{2}\right) \\ &+ \sum_{n=1}^{\infty} \frac{2\left(5 + 3(-1)^{n+1}\right)}{\pi n \sinh(n\pi/2)} \sin\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi y}{2}\right) \\ &+ \sum_{n=1}^{\infty} \frac{10(-1)^{n+1}}{\pi n \sinh(2n\pi)} \sinh(n\pi(2-x)) \sin(n\pi y) \\ &+ \sum_{n=1}^{\infty} \frac{2\left(4 + 3(-1)^{n+1}\right)}{\pi n \sinh(2n\pi)} \sinh(n\pi x) \sin(n\pi y). \end{aligned}$$

Exercise 4. Show that the function

$$f(x,y) = \frac{x^3 - 3xy^2}{(x^2+y^2)^3}$$

is harmonic.

Solution. We must show that $\Delta f = 0$. This is most easily done in polar coordinates, where f becomes

$$f(r,\theta) = \frac{r^3\cos^3\theta - 3r\cos\theta r^2\sin^2\theta}{r^6} = \frac{\cos^3\theta - 3\cos\theta(1-\cos^2\theta)}{r^3} = \frac{4\cos^3\theta - 3\cos\theta}{r^3}.$$

The r partial derivatives are trivial to compute:

$$\begin{aligned} f_r &= \frac{-12\cos^3\theta + 9\cos\theta}{r^4}, \\ f_{rr} &= \frac{48\cos^3\theta - 36\cos\theta}{r^5}. \end{aligned}$$

The θ partials aren't much harder, as long as we use the basic identity $cos^2\theta + \sin^2\theta = 1$ to stick to a single trigonometric function. Indeed, we have

$$\begin{aligned} f_\theta &= \frac{-12\cos^2\theta\sin\theta + 3\sin\theta}{r^3} = \frac{-12(1-\sin^2\theta)\sin\theta + 3\sin\theta}{r^3} = \frac{12\sin^3\theta - 9\sin\theta}{r^3}, \\ f_{\theta\theta} &= \frac{36\sin^2\theta\cos\theta - 9\cos\theta}{r^3} = \frac{36(1-\cos^2\theta)\cos\theta - 9\cos\theta}{r^3} = \frac{-36\cos^3\theta + 27\cos\theta}{r^3}. \end{aligned}$$

Consequently we find that

$$\begin{aligned}
\Delta f &= f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} \\
&= \frac{48\cos^3\theta - 36\cos\theta}{r^5} + \frac{1}{r} \cdot \frac{-12\cos^3\theta + 9\cos\theta}{r^4} + \frac{1}{r^2} \cdot \frac{-36\cos^3\theta + 27\cos\theta}{r^3} \\
&= 0,
\end{aligned}$$

as needed.

Exercise 5. A thin circular metal disk of radius 3 with insulated faces has the temperature along its edge held at 0° in the first quadrant, 50° in the second quadrant, 0° in the third quadrant and 25° in the fourth quadrant. Find the resulting steady-state temperature distribution throughout the disk.

Solution. We need the Fourier series for the 2π-periodic function given on $[0, 2\pi)$ by

$$f(\theta) = \begin{cases} 0 & \text{if } 0 \le \theta < \pi/2, \\ 50 & \text{if } \pi/2 \le \theta < \pi, \\ 0 & \text{if } \pi \le \theta < 3\pi/2, \\ 25 & \text{if } 3\pi/2 \le \theta < 2\pi. \end{cases}$$

We have

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\left(50 \cdot \frac{\pi}{2} + 25 \cdot \frac{\pi}{2}\right) = \frac{75}{4}$$

and for $n \ge 1$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta = \frac{1}{n\pi}\left(-50 \sin\left(\frac{n\pi}{2}\right) - 25 \sin\left(\frac{3n\pi}{2}\right)\right), \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta = \frac{1}{n\pi}\left(50\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right) + 25\left(\cos\left(\frac{3n\pi}{2}\right) - 1\right)\right).
\end{aligned}$$

Therefore the solution to the steady-state heat problem (a Dirichlet problem in this case) is

$$\begin{aligned}
u(r,\theta) = \frac{75}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{3}\right)^n \frac{1}{n} \Bigg(&\left(-50 \sin\left(\frac{n\pi}{2}\right) - 25 \sin\left(\frac{3n\pi}{2}\right)\right) \cos(n\theta) \\
&+ \left(50\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right) + 25\left(\cos\left(\frac{3n\pi}{2}\right) - 1\right)\right) \sin(n\theta) \Bigg).
\end{aligned}$$

Exercise 6. Show that

$$f(x) = \frac{x}{x^2 + x - 2}$$

is analytic at $a = 0$. [*Suggestion:* First find the partial fraction decomposition of f.]

Solution. We have the partial fraction decomposition

$$f(x) = \frac{x}{x^2 + x - 2} = \frac{x}{(x+2)(x-1)} = \frac{1/3}{x-1} + \frac{2/3}{x+2}.$$

Moreover,

$$\frac{1}{x-1} = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

and

$$\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{1-(-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n \quad \text{for } |x/2| < 1.$$

Thus, on the smaller of the two intervals of convergence, namely $|x| < 1$, we have

$$\begin{aligned} f(x) &= \frac{1}{3} \cdot \frac{1}{x-1} + \frac{2}{3} \cdot \frac{1}{x+2} \\ &= -\frac{1}{3}\sum_{n=0}^{\infty} x^n + \frac{2}{3} \cdot \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{3} + \frac{(-1)^n}{3 \cdot 2^n}\right) x^n. \end{aligned}$$

Since this series converges for $|x| < 1$, it has radius at least 1 (it turns out to have radius exactly 1, although we don't need this fact), and since it agrees with f on that same interval, this shows f is analytic at $a = 0$.

Exercise 7. Show that $a = 0$ is an ordinary point of the second order ODE

$$(2 + x^2)y'' - xy' - 3y = 0. \tag{2}$$

Find the recursion relation satisfied by the coefficients of the power series expansion centered at $a = 0$ of any solution to (2), and give a lower bound on its radius of convergence. Find explicit expressions for the coefficients in each of two linearly independent solutions and compute their radii of convergence exactly.

Solution. Putting (2) in standard form we have

$$y'' - \frac{x}{2+x^2}y' - \frac{3}{2+x^2}y = 0.$$

Utilizing the sum of the geometric series we find that

$$\frac{-x}{2+x^2} = \frac{-x}{2} \cdot \frac{1}{1-(-x^2/2)} = \frac{-x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} x^{2n+1}$$

and

$$-\frac{3}{2+x^2} = -\frac{3}{2} \cdot \frac{1}{1-(-x^2/2)} = -\frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{3(-1)^{n+1}}{2^{n+1}} x^{2n}$$

for $|x^2/2| < 1$ or, equivalently, $|x| < \sqrt{2}$. Hence the coefficient functions are analytic at $a = 0$ and therefore $a = 0$ is an ordinary point of (2). In particular, every solution to (2) is analytic at $a = 0$ with radius at least $\sqrt{2}$.

We can therefore write

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and substitute this expression into (2). Distributing, reindexing and collecting common powers of x^n lead to the recursion

$$a_{n+2} = -\frac{(n-3)}{2(n+2)} a_n, \quad n \geq 0.$$

Choosing $a_0 = 1$ and $a_1 = 0$ leads to $a_{2k+1} = 0$ for all $k \geq 0$ and

$$a_2 = \frac{3}{4}, \quad a_4 = \frac{3}{32}, \quad a_{2k} = \binom{2k-3}{k} \frac{3(-1)^k}{2^{3(k-1)}(2k-3)(2k-4)} \quad \text{for } k \geq 3.$$

This give us our first fundamental solution

$$\boxed{y_1 = 1 + \frac{3}{4}x^2 + \frac{3}{32}x^4 + \sum_{k=3}^{\infty} \binom{2k-3}{k} \frac{3(-1)^k x^{2k}}{2^{3(k-1)}(2k-3)(2k-4)}}.$$

To obtain the other fundamental solution we set $a_0 = 0$ and $a_1 = 1$. The first of these tells us that $a_{2k} = 0$ for all $k \geq 0$ and the second that

$$a_3 = \frac{1}{3}, \quad a_5 = 0 \quad \Rightarrow \quad a_{2k+1} = 0 \quad \text{for } k \geq 2.$$

Hence the second fundamental solution is the polynomial

$$\boxed{y_2 = x + \frac{1}{3}x^3}.$$

Because it is a polynomial, which is defined for all x, y_2 has an infinite radius of convergence. As for y_1, we cannot appeal to either of the tests mentioned in class since every other coefficient is 0 (as the series includes only even powers of x). So we just use the generic ratio test applied to the terms of y_1. This gives us

$$\lim_{k\to\infty} \left| \binom{2(k+1)-3}{k+1} \frac{3(-1)^{k+1}x^{2k+2}}{2^{3k}(2(k+1)-3)(2(k+1)-4)} \Big/ \binom{2k-3}{k} \frac{3(-1)^k x^{2k}}{2^{3(k-1)}(2k-3)(2k-4)} \right| =$$
$$\lim_{k\to\infty} \binom{2k-1}{k+1} \Big/ \binom{2k-3}{k} \cdot \frac{|x|^2(2k-3)(2k-4)}{8(2k-1)(2k-2)} = \frac{|x|^2}{8} \lim_{k\to\infty} \frac{(2k-1)!k!(k-3)!}{(k+1)!(k-2)!(2k-3)!} =$$
$$\frac{|x|^2}{8} \lim_{k\to\infty} \frac{(2k-1)(2k-2)}{(k+1)(k-2)} = \frac{|x|^2}{2}.$$

The series for y_1 therefore converges if $|x|^2/2 < 1$ and diverges if $|x|^2/2 > 1$, which means that its radius of convergence is exactly $\sqrt{2}$.

Exercise 8. Find the two values of r for which

$$2x^2y'' + 3xy' + (2x^2 - 1)y = 0$$

has solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0,$$

in which the power series has positive radius of convergence. For each value of r find the recursion relation satisfied by the coefficients a_n and (assuming $a_0 = 1$) the first 5 terms in the series. If possible, find a general expression for a_n in each case.

Solution. Writing

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

substituting this expression into the ODE, distributing, reindexing and collecting common powers of x yields the three equations

$$(2r(r-1) + 3r - 1)a_0 = 0,$$
$$(2(r+1)r + 3(r+1) - 1)a_1 = 0,$$
$$(2(n+r)(n+r-1) + 3(n+r) - 1)\, a_n + 2a_{n-2} = 0, \quad \text{for } n \geq 2.$$

Because $a_0 \neq 0$ the first of these is equivalent to $r(r-1) + 3r - 1 = 0$ or $2r^2 + r - 1 = (2r-1)(r+1) = 0$. Hence $\boxed{r = -1, 1/2}$. If $r = -1$, the second equation becomes $-a_1 = 0$ which means $a_1 = 0$. On the other hand, if $r = 1/2$, it becomes $5a_1 = 0$, again implying that $a_1 = 0$. So, in both cases we have $a_1 = 0$. Solving the third equation for a_n gives us the recursion

$$a_n = \frac{-2}{2(n+r)(n+r-1) + 3(n+r) - 1} a_{n-2} \quad \text{for } n \geq 2.$$

Note that since $a_1 = 0$, this tells us that $a_3 = a_5 = a_7 = \cdots = a_{2k+1} = 0$ for either value of r.

When $r = -1$ the recursion simplifies to

$$a_n = \frac{-2}{n(2n-3)} a_{n-2} \quad \text{for } n \geq 2.$$

After computing the first few terms directly we quickly see that if $a_0 = 1$, then

$$a_{2k} = \frac{(-1)^k}{k! \cdot 1 \cdot 5 \cdot 9 \cdots (4k-3)} \quad \text{for } k \geq 1.$$

We therefore arrive at our first solution:

$$y_1 = x^{-1} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k! \cdot 1 \cdot 5 \cdot 9 \cdots (4k-3)} \right).$$

It is not difficult to use the Ratio Test to show that the power series portion of the solution has an infinite radius of convergence and therefore that y_1 is defined for all $x \neq 0$.

When $r = 1/2$ the recursion instead becomes

$$a_n = \frac{-2}{n(2n+3)} a_{n-2} \quad \text{for } n \geq 2.$$

which, with $a_0 = 1$, eventually leads to the general expression

$$a_{2k} = \frac{(-1)^k}{k! \cdot 7 \cdot 11 \cdot 15 \cdots (4k+3)} \quad \text{for } k \geq 1.$$

This gives us

$$y_2 = x^{1/2} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k! \cdot 7 \cdot 11 \cdot 15 \cdots (4k+3)} \right).$$

Again, the power series factor converges for all x, but $\sqrt{x}$ is only differentiable for $x > 0$, so this is the domain of y_2.

The functions y_1 and y_2 are actually quite interesting. Just for fun, they are shown below, plotted on the interval $(0, 30]$. Can you identify which is which?