Exercise 1. If $(a, b) \neq (0, 0)$, find the general solution to the PDE

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = u.$$

Show that every nonzero solution is unbounded. [*Suggestion:* The "usual" approach will work, but try recognizing the LHS as a directional derivative.]

Exercise 2. Solve the initial value problem

$$\begin{aligned} &x\frac{\partial u}{\partial x} + (t+1)\frac{\partial u}{\partial t} = u, \quad t > 0, \\ &u(x, 0) = f(x). \end{aligned}$$

Exercise 3. Find the Fourier series for the $2p$-periodic function given on the interval $[0, 2p)$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < p, \\ (x-p)^2 & \text{if } p \leq x < 2p. \end{cases}$$

Carefully sketch both the graph of this function and the graph of its Fourier series, for several periods.

Exercise 4. Let g be defined on the interval $[0, 3]$ by

$$g(x) = \begin{cases} 2 & \text{if } 0 \leq x < 1, \\ 3 - x & \text{if } 1 \leq x \leq 3. \end{cases}$$

- **a.** Sketch the even and odd 6-periodic extensions of g, for several periods.
- **b.** Find the sine series expansion of g.
- **c.** Find the cosine series expansion of g.
- **d.** How are parts **a**, **b** and **c** related?

Exercise 5. An ideal elastic string of length L is fixed at its right end, but its left end is attached to a ring that is free to slide up and down a vertical pole. One can show that under these assumptions the tangent line to the free end will always be horizontal. Separate variables and describe the motion of the string at any time $t > 0$, given any initial shape and (vertical) velocity.

Exercise 6. A thin, perfectly insulated, wire is bent into the shape of a circle of radius a.

a. Assuming the temperature, u, in the wire at any point obeys the one-dimensional heat equation (with respect to arc length), show that, in terms of the polar coordinate angle θ, the temperature at any point on the wire satisfies

$$u_t = \left(\frac{c}{a}\right)^2 u_{\theta\theta}.$$

b. Assuming a given initial temperature distribution $f(\theta)$ around the ring, separate variables and find an expression for the temperature at any point in the ring at any later time.

Exercise 7. Solve the following boundary value problem on an $a \times b$ rectangle:

$$\begin{aligned}
&\Delta u = 0, \;\; 0 < x < a, \;\; 0 < y < b, \\
&u(0, y) = u(a, y) = 0, \;\; 0 < y < b, \\
&u_y(x, 0) = -3u(x, 0), \;\; 0 < x < a, \\
&u_y(x, b) = f(x), \;\; 0 < x < a.
\end{aligned}$$

Provide a physical interpretation of the solution.

Exercise 8. Suppose an ideal elastic 1×1 membrane experiences resistance to its motion proportional to its velocity, so that its displacement, u, from equilibrium satisfies the damped wave equation

$$u_{tt} + 2u_t = \frac{1}{\pi^2}\Delta u.$$

Use separation of variables to show that $u \to 0$ exponentially as $t \to \infty$, regardless of the initial conditions.

Exercise 9. Consider *Legendre's differential equation*

$$(1 - x^2)y'' - 2xy' + \mu y = 0, \;\; -1 < x < 1. \tag{1}$$

a. Show that $a = 0$ is an ordinary point of (1).

b. Give a lower bound on the radius of convergence of the power series solutions to (1) of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

c. Find the recursion relation satisfied by the coefficients a_n of the solutions y of part **b**.

d. In applications one often takes $\mu = m(m+1)$, $m \in \mathbb{N}_0$. Show that in this case (1) has a polynomial solution of degree m. Appropriately normalized, these are the *Legendre polynomials*.

Exercise 10. Consider Bessel's equation of order 3:

$$x^2y'' + xy' + (x^2 - 3^2)y = 0, \quad x > 0. \tag{2}$$

a. Show that $a = 0$ is a regular singular point of (2).

b. Determine the value of r for which (2) is *guaranteed* to have a Frobenius solution:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

What is the radius of convergence of the power series factor?

c. For the value of r found in part **b**, find the recursion relation satisfied by the coefficients a_n in the solution y.

d. Write out the first 5 nonzero terms of y, assuming $a_0 = 1$. If possible, find a general formula for the coefficients in the series.