Exercise 1. If $(a, b) \neq (0, 0)$, find the general solution to the PDE

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = u.$$

Show that every nonzero solution is unbounded. [*Suggestion:* The "usual" approach will work, but try recognizing the LHS as a directional derivative.]

Solution. We first proceed in the usual manner, with a linear change of variables:

$$\alpha = mx + ny,$$
$$\beta = rx + sy,$$

with $ms - nr \neq 0$ (we've avoided using the typical a, b, c and d since the first two variables are already taken). As we have seen multiple times, the chain rule tells us that

$$u_x = mu_\alpha + ru_\beta,$$
$$u_y = nu_\alpha + su_\beta,$$

and the original PDE becomes

$$(am + bn)u_\alpha + (ar + bs)u_\beta = u.$$

There are any number of choices to make for m, n, r, s, but we'll go with $r = -b$, $s = a$, $m = a$ and $n = b$. Then $am+bn = a^2+b^2 \neq 0$, $ar+bs = a(-b)+ba = 0$, and $ms-nr = a^2+b^2 \neq 0$. This is therefore a valid change of variables and yields

$$(a^2 + b^2)u_\alpha = u \quad \Rightarrow \quad u_\alpha = \frac{1}{a^2 + b^2}u.$$

In α, this is an exponential growth equation with solution

$$u = C(\beta)\exp\left(\frac{\alpha}{a^2 + b^2}\right).$$

Back substitution gives

$$\boxed{u(x, y) = C(-bx + ay)\exp\left(\frac{ax + by}{a^2 + b^2}\right)}.$$

If u is nonzero, there must be a point (x_0, y_0) so that $C(-bx_0 + ay_0) \neq 0$. Now consider u along the line

$$y = y_0 + \frac{b}{a}(x - x_0).$$

After some simplification we have

$$u\left(x, y_0 + \frac{b}{a}(x - x_0)\right) = C(-bx_0 + ay_0)\exp\left(\frac{(ay_0 - bx_0)b}{a}\right)e^{x/a} = A(x_0, y_0)e^{x/a},$$

with $A(x_0, y_0) \neq 0$. This grows exponentially in x if $a > 0$ and grows exponentially in $-x$ if $a < 0$. So u is unbounded.

This, of course, assumes $a \neq 0$. If $a = 0$ we have

$$u(x, y) = C(-bx)e^{y/b},$$

and a similar argument shows that u grows exponentially in either the y or $-y$ direction.

Exercise 2. Solve the initial value problem

$$\begin{aligned} &x\frac{\partial u}{\partial x} + (t+1)\frac{\partial u}{\partial t} = u, \quad t > 0, \\ &u(x, 0) = f(x). \end{aligned}$$

Solution. We employ the method of characteristics. We begin by parametrizing the initial curve by

$$x(a) = a, \quad t(a) = 0, \quad z(a) = f(a).$$

The characteristic equations of the PDE are then

$$\begin{aligned} &\frac{dx}{ds} = x, \quad \frac{dt}{ds} = t + 1, \quad \frac{dz}{ds} = z, \\ &x(0) = a, \quad t(0) = 0, \quad z(0) = f(a). \end{aligned}$$

The first and last equations are exponential, with solutions

$$x(s) = ae^s, \quad z(s) = f(a)e^s.$$

The equation in t is linear. After rewriting it as

$$\frac{dt}{ds} - t = 1,$$

we introduce the integrating factor $\mu = e^{-s}$, which transforms it into

$$\frac{d}{dt}(te^{-s}) = e^{-s} \quad \Rightarrow \quad te^{-s} = -e^{-s} + C \quad \Rightarrow \quad t = -1 + Ce^s.$$

The initial condition $t(0) = 0$ implies $C = 1$ so that we have

$$t(s) = -1 + e^s.$$

Now we need to express a and s in terms of x and t. From the equation for t we have

$$e^s = t + 1$$

so that the expression for x gives

$$a = xe^{-s} = \frac{x}{t+1}.$$

Plugging these results into the formula for z gives us

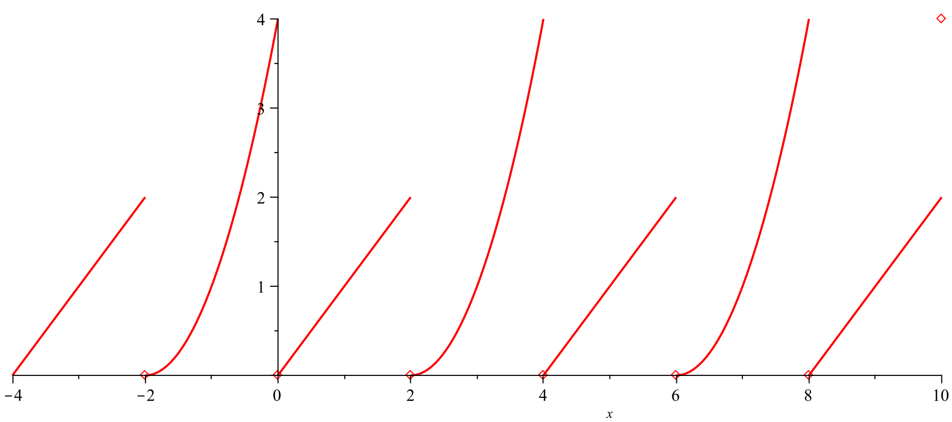
$$\boxed{u(x,t) = z = f\left(\frac{x}{t+1}\right)(t+1)}.$$

Exercise 3. Find the Fourier series for the $2p$-periodic function given on the interval $[0, 2p)$ by

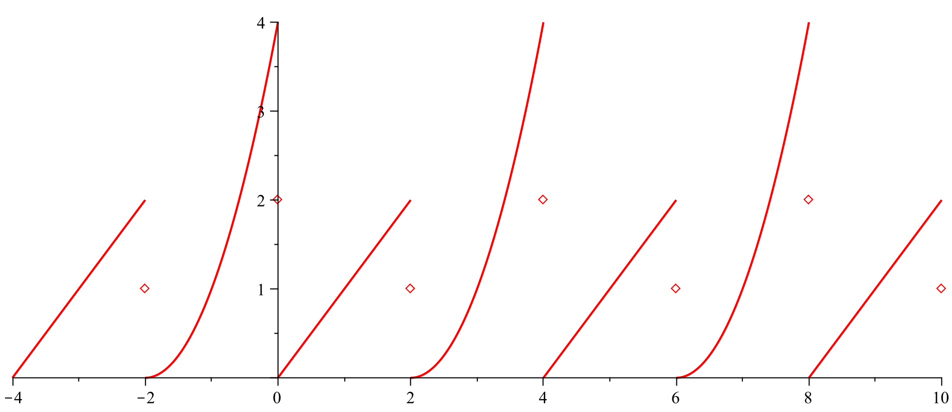
$$f(x) = \begin{cases} x & \text{if } 0 \le x < p, \\ (x-p)^2 & \text{if } p \le x < 2p. \end{cases}$$

Carefully sketch both the graph of this function and the graph of its Fourier series, for several periods.

Solution. The graph of the function is shown below, in the case $p = 2$.



Here is the graph of its Fourier series (the open circles should really be closed):



The function in question is neither even nor odd, so we need to work out both sets of Fourier coefficients. We appeal to Euler's formulae, using the interval $[0, 2p)$ rather than $[-p, p)$ for convenience.

$$
\begin{aligned}
a_0 &= \frac{1}{2p}\left(\int_0^p x\,dx + \int_p^{2p} (x-p)^2\,dx\right) = \frac{2p^2+3p}{12},\\
a_n &= \frac{1}{p}\left(\int_0^p x\cos\left(\frac{n\pi x}{p}\right)\,dx + \int_p^{2p} (x-p)^2\cos\left(\frac{n\pi x}{p}\right)\,dx\right) = \frac{p\,((-1)^n + 2\,p - 1)}{n^2\pi^2},\\
b_n &= \frac{1}{p}\left(\int_0^p x\sin\left(\frac{n\pi x}{p}\right)\,dx + \int_p^{2p} (x-p)^2\sin\left(\frac{n\pi x}{p}\right)\,dx\right)\\
&= -\frac{p\,((n^2\pi^2 + 2\,p)\,(-1)^n + p\,(n^2\pi^2 - 2))}{n^3\pi^3}.
\end{aligned}
$$

This gives us the Fourier series

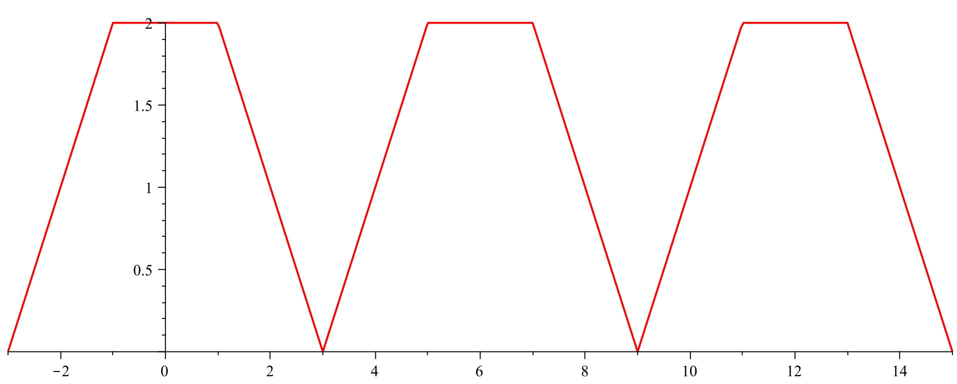
$$
\boxed{
\begin{aligned}
&\frac{2p^2+3p}{12} + \sum_{n=1}^{\infty}\left(\frac{p\,((-1)^n + 2\,p - 1)}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{p}\right)\\
&\qquad + \left(-\frac{p\,((n^2\pi^2 + 2\,p)\,(-1)^n + p\,(n^2\pi^2 - 2))}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{p}\right).
\end{aligned}
}
$$

Exercise 4. Let g be defined on the interval $[0, 3]$ by

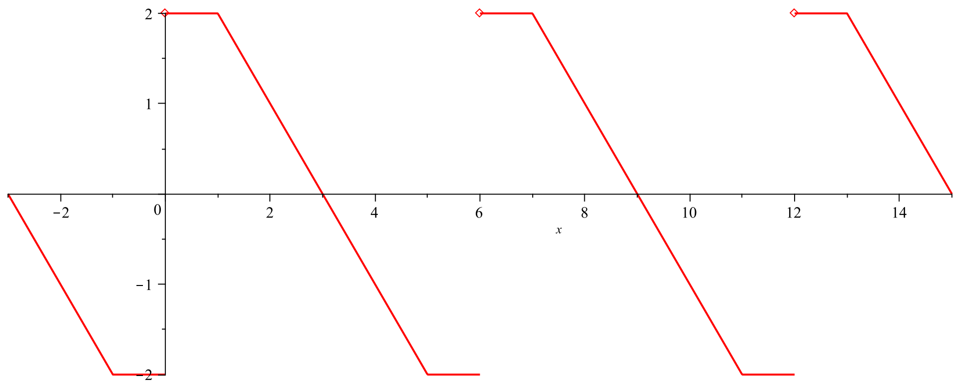
$$
g(x) = \begin{cases} 2 & \text{if } 0 \le x < 1,\\ 3 - x & \text{if } 1 \le x \le 3. \end{cases}
$$

a. Sketch the even and odd 6-periodic extensions of g, for several periods.

Solution. Here's the graph of the even extension, for 3 periods:



And here's the graph of the odd extension, for 3 periods:



b. Find the sine series expansion of g.

Solution. The sine coefficients are given by

$$a_n = \frac{2}{3}\int_0^3 g(x)\sin\left(\frac{n\pi x}{3}\right)\,dx = \frac{4}{n\pi} + \frac{6\,\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2}.$$

Therefore the sine series is

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{4}{n\pi} + \frac{6\,\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{3}\right)}.$$

c. Find the cosine series expansion of g.

Solution. The cosine coefficients are given by

$$a_0 = \frac{1}{3}\int_0^3 g(x)\,dx = \frac{4}{3},$$

$$a_n = \frac{2}{3}\int_0^3 g(x)\cos\left(\frac{n\pi x}{3}\right)\,dx = \frac{6\,\cos\left(\frac{n\pi}{3}\right) + 6\,(-1)^{1+n}}{n^2\pi^2}.$$

which gives us the cosine series

$$\boxed{\frac{4}{3} + \sum_{n=1}^{\infty}\left(\frac{6\,\cos\left(\frac{n\pi}{3}\right) + 6\,(-1)^{1+n}}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{3}\right)}.$$

d. How are parts **a**, **b** and **c** related?

Solution. The sine series expansion of g is simply the Fourier series of its 6-periodic odd extension, whereas the cosine series is the Fourier series of the 6-periodic even extension.

Exercise 5. An ideal elastic string of length L is fixed at its right end, but its left end is attached to a ring that is free to slide up and down a vertical pole. One can show that under these assumptions the tangent line to the free end will always be horizontal. Separate variables and describe the motion of the string at any time $t > 0$, given any initial shape and (vertical) velocity.

Solution. Positioning the string along the interval $[0, L]$ of the x-axis, we need to solve the boundary value problem

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \ \ 0 < x < L, \ \ t > 0, \\ &u_x(0,t) = u(L,t) = 0, \ \ t > 0, \\ &u(x,0) = f(x), \ \ u_t(x,0) = g(x), \ \ 0 < x < L. \end{aligned}$$

We begin by separating variables. Assumer $u(x,t) = X(x)T(t)$. Then u solves the wave equation if and only if $XT'' = c^2 X'' T$ or

$$\frac{T''}{c^2 T} = \frac{X''}{X} = k,$$

a constant, since the two halves of the equation are functions of distinct independent variables. This yields the ODEs

$$\begin{aligned} X'' - kX &= 0, \\ T'' - kc^2 T &= 0. \end{aligned}$$

The (homogeneous) boundary conditions imply that

$$\begin{aligned} 0 = u_x(0,t) = X'(0)T(t) \ &\Rightarrow \ X'(0) = 0, \\ 0 = u(L,t) = X(L)T(t) \ &\Rightarrow \ X(L) = 0, \end{aligned}$$

since we do not want $T \equiv 0$. As this yields a boundary value problem in X that we haven't solved before, we must analyze the sign of the separation constant on a case-by-case basis.

- $k = 0$. In this situation the ODE for X becomes $X'' = 0$ implying $X = ax + b$. The boundary conditions then tell us that

  $$\begin{aligned} 0 &= X'(0) = a, \\ 0 &= X(L) = aL + b = b, \end{aligned}$$

  so that $X \equiv 0$. We may therefore disregard this case.

- $k = \mu^2 > 0$. Here ODE in X becomes $X'' - \mu^2 X = 0$, which has characteristic equation $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$. Therefore $X = a\cosh(\mu x) + b\sinh(\mu x)$ and the boundary conditions tell us that

  $$\begin{aligned} 0 &= X'(0) = \mu(a\sinh 0 + b\cosh 0) = \mu b \ \Rightarrow \ b = 0, \\ 0 &= X(L) = a\cosh(\mu L) + b\sinh(\mu L) = a\cosh(\mu L) = 0 \ \Rightarrow \ a = 0, \end{aligned}$$

  since cosh never vanishes. Once again we conclude that $X \equiv 0$ and move on.

- $k = -\mu^2 < 0$. We now have $X'' + \mu^2 X = 0$ whose characteristic equation has roots $r = \pm i\mu$. Thus $X = a\cos(\mu x) + b\sin(\mu x)$. Now the boundary conditions imply

  $$\begin{aligned} 0 &= X'(0) = \mu(a - \sin 0 + b\cos 0) = \mu b \ \Rightarrow \ b = 0, \\ 0 &= X(L) = a\cos(\mu L) + b\sin(\mu L) = a\cos(\mu L) = 0 \ \Rightarrow \ \cos(\mu L) = 0, \end{aligned}$$

since we do not want $X \equiv 0$. This means

$$\mu L = (2m+1)\frac{\pi}{2} \quad \Rightarrow \quad \mu = \mu_m = \frac{(2m+1)\pi}{2L}, \quad m \in \mathbb{Z}$$

and therefore (taking $a = 1$)

$$X = X_m = \cos\left(\frac{(2m+1)\pi x}{2L}\right), \quad m \in \mathbb{N}_0,$$

where we have discarded the negative values of m since cosine is an even function.
Returning to T, whose equation is now $T'' + c^2\mu_m^2 T = 0$ with characteristic roots $r = \pm ic\mu_m$, we have

$$T = T_m = a_m \cos(c\mu_m t) + b_m \sin(c\mu_m t).$$

We therefore have the normal modes

$$u_m = X_m T_m = \cos\left(\frac{(2m+1)\pi x}{2L}\right)\left(a_m \cos(c\mu_m t) + b_m \sin(c\mu_m t)\right),$$

where

$$\mu_m = \frac{(2m+1)\pi}{2L}, \quad m \in \mathbb{N}_0.$$

Superposition gives the general solution

$$\boxed{u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = \sum_{m=0}^{\infty} \cos\left(\frac{(2m+1)\pi x}{2L}\right)\left(a_m \cos(c\mu_m t) + b_m \sin(c\mu_m t)\right)}.$$

Imposing the two initial conditions tells us that

$$f(x) = u(x,0) = \sum_{m=0}^{\infty} a_m \cos\left(\frac{(2m+1)\pi x}{2L}\right),$$

$$g(x) = u_t(x,0) = \sum_{m=0}^{\infty} \mu_m b_m \cos\left(\frac{(2m+1)\pi x}{2L}\right).$$

it is easy to verify that the functions $\cos(\mu_m x)$ are orthogonal on the interval $[0, L]$, that is

$$\langle \cos(\mu_m x), \cos(\mu_n x)\rangle = \int_0^L \cos(\mu_m x)\, \cos(\mu_n x)\, dx = 0.$$

Hence we may extract the coefficients of both series as ratios of inner products:

$$\boxed{\begin{aligned} a_m &= \frac{\langle f(x), \cos(\mu_m x)\rangle}{\langle \cos(\mu_m x), \cos(\mu_m x)\rangle} = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2m+1)\pi x}{L}\right) dx, \\ \mu_m b_m &= \frac{\langle g(x), \cos(\mu_m x)\rangle}{\langle \cos(\mu_m x), \cos(\mu_m x)\rangle} \quad \Rightarrow \quad b_m = \frac{2}{\mu_m L}\int_0^L g(x)\cos\left(\frac{(2m+1)\pi x}{L}\right) dx. \end{aligned}}$$

This gives the complete solution to the problem at hand.

Exercise 6. A thin, perfectly insulated, wire is bent into the shape of a circle of radius a.

a. Assuming the temperature, u, in the wire at any point obeys the one-dimensional heat equation (with respect to arc length), show that, in terms of the polar coordinate angle θ, the temperature at any point on the wire satisfies

$$u_t = \left(\frac{c}{a}\right)^2 u_{\theta\theta}.$$

Solution. If we center the circle at the origin and let x denotes the arc length of a segment of the circle measured counterclockwise from the horizontal axis, then $x = a\theta$. The chain rule readily tells us that

$$u_{xx} = \frac{1}{a^2} u_{\theta\theta}$$

so that the usual heat equation $u_t = c^2 u_{xx}$ becomes

$$u_t = \left(\frac{c}{a}\right)^2 u_{\theta\theta},$$

as claimed.

b. Assuming a given initial temperature distribution $f(\theta)$ around the ring, separate variables and find an expression for the temperature at any point in the ring at any later time.

Solution. We write $u(\theta, t) = \Theta(\theta)T(t)$ and plug into the modified PDE to get

$$\Theta T' = \left(\frac{c}{a}\right)^2 \Theta'' T \quad \Rightarrow \quad \frac{\Theta''}{a^2\Theta} = \frac{T'}{c^2 T} = k,$$

or, equivalently,

$$\begin{aligned} \Theta'' - a^2 k\Theta &= 0, \\ T' - c^2 kT &= 0. \end{aligned}$$

Because the temperature at any point on the circle has to remain the same after we've made one complete revolution, we have the "boundary condition" that Θ must be 2π-periodic. In order to achieve periodicity in general this requires

$$-a^2 k > 0.$$

Since this forces $k < 0$, we write $k = -\mu^2$ so that the ODE in Θ is $\Theta'' + a^2\mu^2\Theta$ and

$$\Theta = A\cos(a\mu\theta) + B\sin(a\mu\theta).$$

One can show that the only way this can have period 2π is if $a\mu = n$ for some $n \in \mathbb{Z}$, or

$$\mu = \mu_n = \frac{n}{a} \quad \Rightarrow \quad \Theta = \Theta_n = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n \in \mathbb{N}_0,$$

since we can use the symmetry properties of sine and cosine to absorb the sign of n into the constants.

Returning to T, for each $n \in \mathbb{N}_0$ we have the equation

$$T' + c^2\mu_n^2 T = 0 \;\Rightarrow\; T' = -c^2\mu_n^2 T \;\; T = T_n = e^{-c^2\mu_n^2 t} = e^{-c^2 n^2 t/a^2},$$

where we have taken the multiplicative constant to be 1 for convenience. This gives us the normal modes

$$u_n(x,t) = X_n T_n = e^{-c^2 n^2 t/a^2}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right).$$

Note that when $n = 0$ we simply have

$$u_0(x,t) = A_0.$$

Hence superposition yields the general solution

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-c^2 n^2 t/a^2}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)}.$$

Finally, to determine A_n and B_n we impose the initial condition

$$f(\theta) = u(\theta,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta),$$

which is simply the 2π-periodic Fourier series for f. According to Euler's formulae

$$\boxed{\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta, \\ A_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta, \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta, \end{aligned}}$$

completing the solution.

Exercise 7. Solve the following boundary value problem on an $a \times b$ rectangle:

$$\begin{aligned} &\Delta u = 0, \;\; 0 < x < a, \;\; 0 < y < b, \\ &u(0,y) = u(a,y) = 0, \;\; 0 < y < b, \\ &u_y(x,0) = -3u(x,0), \;\; 0 < x < a, \\ &u_y(x,b) = f(x), \;\; 0 < x < a. \end{aligned}$$

Provide a physical interpretation of the solution.

Solution. The function u represents the steady-state temperature that results in a laterally insulated rectangular metal plate when the temperatures on the left and right edges are held constantly at 0°, the top edge is subject to a (temporally) constant heat flux at each point, and the bottom edge radiates heat at a rate proportional to the current temperature there.

To solve the problem we, perhaps not surprisingly, separate variables. Writing $u(x, y) = X(x)Y(y)$ the Laplace equation $\Delta u = 0$ becomes $X''Y + XY'' = 0$ or, after dividing by XY,

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = k,$$

since the two sides of the latter equation are functions of distinct independent variables. This gives the pair of ODEs

$$\begin{aligned} X'' - kX &= 0, \\ Y'' + kY &= 0. \end{aligned}$$

The homogeneous boundary conditions yield

$$\begin{aligned} &0 = u(0, y) = X(0)Y(y) \quad \Rightarrow \quad X(0) = 0, \\ &0 = u(a, y) = X(a)Y(y) \quad \Rightarrow \quad X(a) = 0, \\ &0 = u_y(x, 0) + 3u(x, 0) = X(x)Y'(0) + 3X(x)Y(0) \quad \Rightarrow \quad Y'(0) + 3Y(0) = 0, \end{aligned}$$

since we do not want $X \equiv 0$ or $Y \equiv 0$.

We've solved the boundary value problem in X numerous times. The solutions are

$$X = X_n = \sin(\mu_n x), \quad n \in \mathbb{N},$$

where $\mu_n = n\pi/a$ and $k = -\mu_n^2$. Returning to Y yields the ODE $Y'' - \mu_n^2 Y = 0$ with solutions

$$Y_n = A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y).$$

Imposing the condition $Y'(0) + 3Y(0) = 0$ this becomes

$$\mu_n B_n + 3A_n = 0 \quad \Rightarrow \quad A_n = -\frac{\mu_n B_n}{3}$$

so that

$$Y_n = B_n \left(-\frac{\mu_n}{3} \cosh(\mu_n y) + \sinh(\mu_n y) \right).$$

We therefore have the modes

$$u_n(x, y) = X_n Y_n = B_n \sin(\mu_n x) \left(-\frac{\mu_n}{3} \cosh(\mu_n y) + \sinh(\mu_n y) \right), \quad \boxed{\mu_n = \frac{n\pi}{a}}, \quad n \in \mathbb{N}.$$

Consequently the general solution is given by the superposition

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x) \left(-\frac{\mu_n}{3} \cosh(\mu_n y) + \sinh(\mu_n y) \right)}.$$

Hence

$$u_y(x, y) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x) \left(-\frac{\mu_n^2}{3} \sinh(\mu_n y) + \mu_n \cosh(\mu_n y) \right)$$

and the final boundary condition becomes

$$f(x) = u_y(x,b) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x)\left(-\frac{\mu_n^2}{3}\sinh(\mu_n b) + \mu_n \cosh(\mu_n b)\right).$$

This is just the $2a$-periodic sine expansion of $f(x)$, with coefficients a bit jumbled. That is

$$B_n\left(-\frac{\mu_n^2}{3}\sinh(\mu_n b) + \mu_n \cosh(\mu_n b)\right) = \frac{1}{a}\int_0^a f(x)\sin(\mu_n x)\,dx$$

so that

$$\boxed{B_n = \frac{1}{a\left(-\frac{\mu_n^2}{3}\sinh(\mu_n b) + \mu_n \cosh(\mu_n b)\right)}\int_0^a f(x)\sin(\mu_n x)\,dx}.$$

This completes the solution.

Exercise 8. Suppose an ideal elastic 1×1 membrane experiences resistance to its motion proportional to its velocity, so that its displacement, u, from equilibrium satisfies the damped wave equation

$$u_{tt} + 2u_t = \frac{1}{\pi^2}\Delta u.$$

Use separation of variables to show that $u \to 0$ exponentially as $t \to \infty$, regardless of the initial conditions.

Solution. Guess what? We separate variables. Write $u(x,y,t) = X(x)Y(y)T(t)$ and plug into the damped wave equation to obtain

$$XYT'' + 2XYT' = \frac{1}{\pi^2}(X''YT + XY''T) \;\;\Rightarrow\;\; \pi^2\left(\frac{T''}{T} + 2\frac{T'}{T}\right) = \frac{X''}{X} + \frac{Y''}{Y} = A,$$

since in the final equation the LHS depends only on t while the RHS depends only on x and y. From the second half of this equation we obtain

$$\frac{X''}{X} = A - \frac{Y''}{Y} = B,$$

again for similar reasoning. Setting $C = A - B$ and clearing denominators we obtain the ODEs

$$\begin{aligned}
&X'' - BX = 0,\\
&Y'' - CB = 0,\\
&T'' + 2T' - \frac{B+C}{\pi^2}T = 0.
\end{aligned}$$

With the edges of the membrane fixed, we have the boundary conditions

$$\begin{aligned}
0 = X(x)Y(0)T(t) \;\;&\Rightarrow\;\; Y(0) = 0,\\
0 = X(x)Y(1)T(t) \;\;&\Rightarrow\;\; Y(1) = 0,\\
0 = X(0)Y(y)T(t) \;\;&\Rightarrow\;\; X(0) = 0,\\
0 = X(1)Y(y)T(t) \;\;&\Rightarrow\;\; X(1) = 0,
\end{aligned}$$

where, once again, we have used the facts that we *do not* want $X \equiv 0$, $Y \equiv 0$, $T \equiv 0$.

We have yet again found familiar boundary value problems, in X and Y. The solutions are

$$\begin{aligned} X &= X_m = \sin(m\pi x), \quad m \in \mathbb{N}, \\ Y &= Y_n = \sin(n\pi y), \quad n \in \mathbb{N}, \end{aligned}$$

with $B = -(m\pi)^2$, $C = -(n\pi)^2$. We let

$$\lambda_{mn} = \sqrt{-\frac{B+C}{\pi^2}} = \sqrt{(m^2+n^2)}.$$

so that the T ODE now reads

$$T'' + 2T' + \lambda_{mn}^2 T = 0.$$

This is a constant coefficient equation and the roots of its characteristic polynomial are

$$\begin{aligned} \frac{-2 \pm \sqrt{4 - 4\lambda_{mn}^2}}{2} &= -1 \pm \sqrt{1 - \lambda_{mn}^2} = -1 \pm \sqrt{1 - m^2 - n^2} \\ &= -1 \pm i\sqrt{m^2 + n^2 - 1} = -1 \pm i\sqrt{\lambda_{mn}^2 - 1}, \end{aligned}$$

since $m, n \geq 1$ implies that $1 - m^2 - n^2 \leq -1 < 0$. Hence

$$T = T_{mn} = e^{-t}\left(A_{mn}\cos\left(t\sqrt{\lambda_{mn}^2 - 1}\right) + B_{mn}\sin\left(t\sqrt{\lambda_{mn}^2 - 1}\right)\right).$$

We finally conclude that the normal modes are

$$u_{mn}(x,y,t) = X_m Y_n T_{mn} = \sin(m\pi x)\sin(n\pi y)e^{-t}\left(A_{mn}\cos\left(t\sqrt{\lambda_{mn}^2 - 1}\right) + B_{mn}\sin\left(t\sqrt{\lambda_{mn}^2 - 1}\right)\right).$$

Superimposing gives the general solution

$$\boxed{u(x,y,t) = e^{-t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(m\pi x)\sin(n\pi y)\left(A_{mn}\cos\left(t\sqrt{\lambda_{mn}^2 - 1}\right) + B_{mn}\sin\left(t\sqrt{\lambda_{mn}^2 - 1}\right)\right)}.$$

The series factor is oscillatory in t, but the e^{-t} factor out front drives it to zero exponentially as $t \to \infty$, as claimed.

Exercise 9. Consider *Legendre's differential equation*

$$(1 - x^2)y'' - 2xy' + \mu y = 0, \quad -1 < x < 1. \tag{1}$$

a. Show that $a = 0$ is an ordinary point of (1).

Solution. Rewriting (1) in standard form gives

$$y'' - \frac{2x}{1-x^2}y' + \frac{\mu}{1-x^2} = 0.$$

Since

$$p(x) = -\frac{2x}{1-x^2} = -2x\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} -2x^{2n+1}, \quad |x^2| < 1 \;\Leftrightarrow\; |x| < 1$$

and

$$q(x) = \frac{\mu}{1-x^2} = \mu\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} \mu x^{2n}, \quad |x^2| < 1 \;\Leftrightarrow\; |x| < 1,$$

both p and q are analytic at $a = 0$ with radii of convergence (at least) 1.

b. Give a lower bound on the radius of convergence of the power series solutions to (1) of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Solution. As noted above, both p and q are analytic at $a = 0$ with radii of convergence (at least) 1. Hence every solution to (1) is also given by a power series centered at $a = 0$ with at least radius of convergence at least 1.

c. Find the recursion relation satisfied by the coefficients a_n of the solutions y of part **b**.

Solution. Substituting

$$y = \sum_{n=0}^{\infty} a_n x^n$$

into (1) yields

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{x-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + \mu\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + \mu\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + \mu\sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 + \mu a_0) + (6a_3 - 2a_1 + \mu a_1)x + \sum_{n=2}^{\infty}\left((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \mu a_n\right)x^n = 0.$$

The identity principle gives

$$2a_2 + \mu a_0 = 0 \;\Rightarrow\; a_2 = -\frac{\mu a_0}{2},$$

$$6a_3 - 2a_1 + \mu a_1 = 0 \;\Rightarrow\; a_3 = \frac{(2-\mu)a_1}{6},$$

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \mu a_n = 0 \;\Rightarrow\; \boxed{a_{n+2} = \frac{(n^2+n-\mu)a_n}{(n+2)(n+1)}},$$

the final recursion holding for $n \geq 0$ by direct computation.

d. In applications one often takes $\mu = m(m+1)$, $m \in \mathbb{N}_0$. Show that in this case (1) has a polynomial solution of degree m. Appropriately normalized, these are the *Legendre polynomials*.

Solution. When $\mu = m(m+1)$, the recursion of part **c** becomes (after some simplification)

$$a_{n+2} = \frac{(n-m)(n+m+1)a_n}{(n+2)(n+1)}.$$

Suppose m is even and take $a_0 = 1$ and $a_1 = 0$ we find that

$$a_3 = \frac{(1-m)(m+2)a_1}{3 \cdot 2} = 0 \;\Rightarrow\; a_5 = \frac{(3-m)(m+4)a_3}{5 \cdot 4} = 0 \cdots \Rightarrow \quad a_{2k+1} = 0 \text{ for } k \geq 0.$$

and likewise a_0 determines a_2, which determines a_4, which determines a_6, etc. Notice that

$$a_{m+2} = \frac{(m-m)(m+m+1)a_m}{(m+2)(m+1)} = 0 \;\Rightarrow\; a_{2k} = 0 \text{ for } k \geq m+1.$$

Since we have already seen that the odd coefficients vanish, this means that our solution looks like

$$y = a_0 + a_2x^2 + a_4x^4 + \cdots + a_mx^m,$$

i.e. is a polynomial of degree m. If m is odd and we take $a_0 = 0$ and $a_1 = 1$ something entirely similar happens.

Exercise 10. Consider Bessel's equation of order 3:

$$x^2y'' + xy' + (x^2 - 3^2)y = 0, \;\; x > 0. \tag{2}$$

a. Show that $a = 0$ is a regular singular point of (2).

Solution. We put (2) in standard form obtaining

$$y'' + \frac{1}{x}y' + \left(1 - \frac{9}{x^2}\right) y = 0$$

so that $p(x) = 1/x$ and $q(x) = 1 - 9/x^2$. Since these aren't even defined at $x = 0$ they cannot be analytic at $a = 0$, so we definitely have a singular point. However $xp(x) = 1$ and $x^2q(x) = x^2 - 9$ are power seres centered at $a = 0$ with infinite radius of convergence (they are simply polynomials), making the singular point regular.

b. Determine the value of r for which (2) is *guaranteed* to have a Frobenius solution:

$$y = x^r \sum_{n=0}^{\infty} a_nx^n, \;\; a_0 \neq 0.$$

What is the radius of convergence of the power series factor?

Solution. From above we have

$$p_0 = \lim_{x\to 0} xp(x) = 1,$$
$$q_0 = \lim_{x\to 0} x^2 q(x) = -9,$$

so that the indicial equation is

$$r^2 + (1-1)r - 9 = 0, \quad \Rightarrow \quad r = \pm 3.$$

Because $3-(-3) = 6$ is an integer, only the larger value, namely $\boxed{r=3}$, is guaranteed to yield a Frobenius-type solution. Because $xp(x)$ and $x^2q(x)$ have infinite radius of convergence, so, too, does the power series factor of any Frobenius solution.

c. For the value of r found in part **b**, find the recursion relation satisfied by the coefficients a_n in the solution y.

Solution. Taking

$$y = x^3 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+3} = \sum_{n=3}^{\infty} a_{n-3} x^n, \quad a_0 \neq 0,$$

and substituting into (2) we find that

$$x^2 \sum_{n=3}^{\infty} n(n-1)a_{n-3}x^{n-2} + x\sum_{n=3}^{\infty} na_{n-3}x^{n-1} + x^2\sum_{n=3}^{\infty} a_{n-3}x^n - 9\sum_{n=3}^{\infty} a_{n-3}x^n = 0$$
$$\sum_{n=3}^{\infty} n(n-1)a_{n-3}x^n + \sum_{n=3}^{\infty} na_{n-3}x + \sum_{n=3}^{\infty} a_{n-3}x^{n+2} - 9\sum_{n=3}^{\infty} a_{n-3}x^n = 0$$
$$\sum_{n=3}^{\infty} n(n-1)a_{n-3}x^n + \sum_{n=3}^{\infty} na_{n-3}x + \sum_{n=5}^{\infty} a_{n-5}x^n - 9\sum_{n=3}^{\infty} a_{n-3}x^n = 0$$
$$(6a_0 + 3a_0 - 9a_0)x^3 + (12a_1 + 4a_1 - 9a_1)x^4 + \sum_{n=5}^{\infty} \left((n(n-1) + n - 9)a_{n-3} + a_{n-5}\right)x^n = 0.$$

Appealing to the identity principle to equate every coefficient to zero, the first term cancels out (as it should), the second yields $7a_1 = 0$, implying $\boxed{a_1 = 0}$, and the remaining terms tell us that

$$(n(n-1) + n - 9)a_{n-3} + a_{n-5} = 0 \quad \Rightarrow \quad \boxed{a_{n-3} = \frac{-a_{n-5}}{n^2-9} \text{ for } n \geq 5}.$$

d. Write out the first 5 nonzero terms of y, assuming $a_0 = 1$. If possible, find a general formula for the coefficients in the series.

Solution. Since $a_1 = 0$ and the recursion of **c** expresses each coefficient as a multiple of the coefficient two indices earlier, we immediately have $a_3 = a_5 = a_7 = \cdots = a_{2k+1} = 0$ for $k \geq 0$. On the other hand, with $a_0 = 1$ we find that (since $n^2 - 9 = (n-3)(n+3)$)

$$a_2 = \frac{-1}{(5-3)(5+3)} = \frac{-1}{2\cdot 8} \quad \Rightarrow \quad a_4 = \frac{-a_2}{4\cdot 10} = \frac{1}{2\cdot 4\cdot 8\cdot 10} \quad \Rightarrow$$
$$a_6 = \frac{-a_4}{6\cdot 12} = \frac{-1}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} \quad \Rightarrow \quad a_8 = \frac{-a_6}{8\cdot 14} = \frac{1}{2\cdot 4\cdot 6\cdot 8\cdot 8\cdot 10\cdot 12\cdot 14}$$

so that in general we have

$$a_{2k} = \frac{(-1)^k}{2 \cdot 4 \cdots (2k) \cdot 8 \cdot 10 \cdots (2k+6)} = \frac{(-1)^k 2 \cdot 4 \cdot 6}{(2 \cdot 4 \cdots (2k))\,(2 \cdot 4 \cdots (2k+6))}$$
$$= \frac{(-1)^k 48}{2^k k! 2^{k+3}(k+3)!} = \frac{(-1)^k 6}{2^{2k} k!(k+3)!} \text{ for } k \geq 0.$$

Hence the Frobenius solution is

$$\boxed{y = x^3 \sum_{k=0}^{\infty} \frac{(-1)^k 6 x^{2k}}{2^{2k} k!(k+3)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 6 x^{2k+3}}{2^{2k} k!(k+3)!}}.$$

If all we want are the first 5 nonzero terms, then we have

$$\boxed{y = x^3 - \frac{x^5}{16} + \frac{x^7}{640} - \frac{x^9}{46080} + \frac{x^{11}}{5160960} - \cdots}.$$