The Vibrating String Subject to an External Force

R. C. Daileda

February 20, 2018

Under the assumption that the only force acting on an ideal string is its internal tension, we showed that its displacement $u(x,t)$ from equilibrium at position x and time t satisfied the one-dimensional wave equation,

$$u_{tt} = c^2 u_{xx}. \tag{1}$$

In class a student posed the following question: what would happen if we subjected the string to gravity? More generally one might wonder what would happen if the string were subject to a constant force F (per unit length) acting in the u-direction. According to Newton's second law, the resulting acceleration at any point on the string would be $a = F/\rho$, where ρ is the (constant) linear mass density of the string. Since the wave equation gives the acceleration due to the forces internal to the string, the net acceleration of the string would then be

$$u_{tt} = c^2 u_{xx} + a. \tag{2}$$

Now suppose that the string has fixed endpoints at $x = 0$ and $x = L$ so that we have the boundary values

$$u(0,t) = u(L,t) = 0, \ \ t > 0. \tag{3}$$

In addition, suppose we are given the initial shape and velocity of the string:

$$\begin{aligned} u(x,0) &= f(x), \ \ 0 < x < L, \\ u_t(x,0) &= g(x), \ \ 0 < x < L. \end{aligned} \tag{4}$$

We would like to solve the modified wave equation (2) subject to these conditions.

We suspect that instead of the equilibrium position of the string being $u \equiv 0$, the external force creates a new equilibrium, and that the string will oscillate about that position. To find the new equilibrium we simply observe that it must be a solution of (2) with fixed end points that satisfies $u_t \equiv 0$, i.e. $u = u(x)$. Plugging this into (2) we obtain $u'' = -a/c^2$ so that $u = -ax^2/2c^2 + Bx + C$. Since $0 = u(0)$ we immediately obtain $C = 0$. And since $0 = u(L)$ we find that $B = aL/2c^2$. Hence the equilibrium solution is

$$u = -\frac{ax^2}{2c^2} + \frac{aLx}{2c^2} = \frac{ax}{2c^2}(L - x).$$

This is a downward opening parabola if $a > 0$ and an upward opening parabola if $a < 0$, which should agree with our intuition.

Sticking to our idea that the solutions to the modified BVP (2)-(4) are simply oscillations about the new equilibrium, to extract these oscillations we suppose that u is a solution to the new BVP and subtract the new equilibrium from it. Specifically, let

$$v(x,t) = u(x,t) - \frac{ax}{2c^2}(L-x) = u(x,t) + \frac{ax(x-L)}{2c^2}. \tag{5}$$

Note that $v(0,t) = u(0,t) + 0 = 0$ and $v(L,t) = u(L,t) + 0 = 0$ and

$$\begin{aligned} v_{tt} &= u_{tt}, \\ v_{xx} &= u_{xx} + \frac{a}{c^2}, \end{aligned}$$

so that

$$v_{tt} = u_{tt} = c^2 u_{xx} + a = c^2 v_{xx}.$$

That is, v solves the original wave equation (1) and has fixed endpoints at $x = 0$ and $x = L$.

Now let's look at the initial conditions v must satisfy. In light of (4) we have

$$\begin{aligned} v(x,0) &= f(x) + \frac{ax(x-L)}{2c^2}, \quad 0 < x < L, \\ v_t(x,0) &= g(x), \quad 0 < x < L. \end{aligned}$$

Hence v is given by

$$v(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(\widetilde{b_n}\cos\left(\frac{cn\pi t}{L}\right) + \widetilde{b_n^*}\sin\left(\frac{cn\pi t}{L}\right)\right),$$

where

$$\begin{aligned} \widetilde{b_n} &= \frac{2}{L}\int_0^L \left(f(x) + \frac{ax(x-L)}{2c^2}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx + \frac{a}{Lc^2}\int_0^L (x^2 - Lx)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx + \frac{a}{Lc^2}\cdot\frac{2L^3((-1)^n - 1)}{n^3\pi^3} \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx + \frac{2aL^2((-1)^n - 1)}{c^2n^3\pi^3} \\ &= b_n + \frac{2aL^2((-1)^n - 1)}{c^2n^3\pi^3}, \end{aligned}$$

where the b_n are the coefficients in the $2L$-periodic sine expansion of $f(x)$, and

$$\frac{cn\pi}{L}\widetilde{b_n^*} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad \Rightarrow \quad \widetilde{b_n^*} = \frac{L}{cn\pi}b_n',$$

where the b_n' are the coefficients in the $2L$-periodic sine expansion of $g(x)$.

Returning to equation (5), we find that the solution to the forced vibrating string problem with fixed endpoints (2)-(4) is given by

$$\begin{aligned} u(x,t) &= v(x,t) - \frac{ax(x-L)}{2c^2} \\ &= \frac{ax(L-x)}{2c^2} + \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left(\left(b_n + \frac{2aL^2((-1)^n - 1)}{c^2n^3\pi^3}\right)\cos\left(\frac{cn\pi t}{L}\right) + \frac{L}{cn\pi}b_n'\sin\left(\frac{cn\pi t}{L}\right)\right), \end{aligned}$$

in which the coefficients b_n and b_n' are as described above.