Modern Algebra
Spring 2019

Assignment 10.1
Due April 10

Exercise 1. If $\sigma \in S_n$ and $(i_1\, i_2\, \cdots\, i_r) \in S_n$ is an r-cycle, show that

$$\sigma(i_1\, i_2\, \cdots\, i_r)\sigma^{-1} = (\sigma(i_1)\, \sigma(i_2)\, \cdots\, \sigma(i_r)).$$

Use this and the previous exercise to prove that all r-cycles are conjugate in S_n. [*Suggestion:* Prove the two permutations are equal by evaluating them both at every point, noting that every $j \in I_n$ can be written $j = \sigma(i)$ for some $i \in I_n$.]

Given $\sigma \in S_n$, write $\sigma = c_1 c_2 \dots c_k$, where the c_j are disjoint cycles and every $i \in I_n$ occurs in exactly one c_j (we include 1-cycles). Reorder the cycles in σ so that $\ell(c_1) \geq \ell(c_2) \geq \cdots \geq \ell(c_k)$, where $\ell(c)$ denotes the length of a cycle c. Finally, let

$$\lambda(\sigma) = (\ell(c_1), \ell(c_2), \cdots, \ell(c_k)).$$

Exercise 2. Prove that $\lambda(\sigma)$ is a *partition* of n, i.e. that the sum of the entries of $\lambda(\sigma)$ is n.

Exercise 3. Prove that $\lambda(\tau\sigma\tau^{-1}) = \lambda(\sigma)$ for all $\tau \in S_n$. [*Suggestion:* Write σ as a product of disjoint cycles and apply Exercise 1.]

Exercise 4. Prove that the converse of the result in the previous exercise also holds. That is, show that if $\tau \in S_n$ and $\lambda(\sigma) = \lambda(\tau)$, then σ and τ are conjugate in S_n. [*Suggestion:* If you write down the cycle decompositions of σ and τ, and appeal to Exercise 1, you can easily construct a permutation that conjugates σ into τ.]