Modern Algebra
Spring 2019

Assignment 10.2
Due April 10

Exercise 1. Let G be a group and let G' denote its commutator subgroup. Set $G^{(1)} = G'$ and recursively define $G^{(i+1)} = (G^{(i)})'$. Suppose we have descending a sequence of subgroups

$$\cdots \lhd G_2 \lhd G_1 \lhd G_0 = G \tag{1}$$

so that G_i/G_{i+1} is abelian for all i. Prove that $G^{(i)} < G_i$ for all i. [*Suggestion:* Induct. Use the fact that, given $H \lhd G$, G/H is abelian if and only if $G' < H$.]

Exercise 2. Recall that G is *solvable* provided $G^{(r)} = \{e\}$ for some $r \geq 1$. Prove that G is solvable if and only if there is a sequence (1), with abelian factors, that eventually terminates with the trivial group. [*Suggestion:* For one direction, use the preceding exercise.]

Exercise 3. Let G be a finite group with subgroups H and N. If N is normal in G, prove that $[H : H \cap N]$ divides $[G : N]$. [*Suggestion:* This follows almost immediately from the Second Isomorphism Theorem and the Correspondence Principle. Alternatively, one can use the First Isomorphism Theorem to find a monomorphism $H/(H \cap N) \hookrightarrow G/N$.]

Exercise 4. Let H be a finite group of odd order. Given $x \in H$, let $T_x \in \mathrm{Perm}(H)$ denote left translation by x. Prove that T_x is even. [*Suggestion:* Recall that the mapping $x \mapsto T_x$ is a monomorphism $H \hookrightarrow \mathrm{Perm}(H) \cong S_n$, where $n = |H|$. Apply the preceding exercise with $G = S_n$ and $N = A_n$. It's also possible to show that every cycle of T_x has odd length.]